

$\frac{SO(2N)}{U(N)}$ Riccati-Hartree-Bogoliubov Equation Based on the $SO(2N)$ Lie Algebra of the Fermion Operators

Seiya NISHIYAMA* and João da PROVIDÊNCIA†

Centro de Física, Departamento de Física,
Universidade de Coimbra, P-3004-516 Coimbra, Portugal

Dedicated to the Memory of Hideo Fukutome

February 18, 2015

Abstract

In this paper we present the induced representation of $SO(2N)$ canonical transformation group and introduce $\frac{SO(2N)}{U(N)}$ coset variables. We give a derivation of the time dependent Hartree-Bogoliubov (TDHB) equation on the Kähler coset space $\frac{G}{H} = \frac{SO(2N)}{U(N)}$ from the Euler-Lagrange equation of motion for the coset variables. The TDHB wave function represents the TD behavior of Bose condensate of fermion pairs. It is a good approximation for the ground state of the fermion system with a pairing interaction, producing the spontaneous Bose condensation. To describe the classical motion on the coset manifold, we start from the local equation of motion. This equation becomes a Riccati-type equation. After giving a simple two-level model and a solution for a coset variable, we can get successfully a general solution of TDRHB equation for the coset variables. We obtain the Harish-Chandra decomposition for the $SO(2N)$ matrix based on the nonlinear Möbius transformation together with the geodesic flow on the manifold.

Keywords: Kaehler manifold; Hartree-Bogoliubov theory; $SO(2N)$ Lie algebra;
time dependent Riccati-Hartree-Bogoliubov equation

Mathematics Subject Classification 2010: 81Rxx, 81R05, 81Vxx, 81V35

*Corresponding author. E-mail address: seikoceu@khe.biglobe.ne.jp, nishiyama@teor.fis.uc.pt

†E-mail address: providencia@teor.fis.uc.pt

1 Introduction

The supersymmetric (SUSY) extension of the nonlinear σ -model was first given by Zumino under the introduction of scalar fields on a Kähler manifold [1]. The extended σ -model defined on symmetric spaces have been intensively studied in modern elementary particle physics, superstring theory and supergravity theory [2]. While the Hartree-Bogoliubov theory (HBT) [3] has been regarded as the standard approximation in the theory of fermion systems [4, 5]. In HBT a HB wave function (WF) represents a Bose condensate of fermion pairs. From the Lie-algebraic viewpoint, fermion pair operators form a $SO(2N)$ Lie algebra and contain a $U(N)$ Lie algebra as a subalgebra (N : Number of fermion states). The $SO(2N)$ and $U(N)$ mean the special orthogonal group of $2N$ dimensions and unitary group of N dimensions, respectively. One can give the exact coherent state representation (CS rep) of a fermion system [6].

A consistent coupling of gauge- and matter superfields to SUSY σ -models on Kähler coset space has been given by van Holten et al. They have presented a mathematical tool of constructing a Killing potential and have applied it to the explicit construction of a SUSY σ -model on the coset space $\frac{SO(2N)}{U(N)}$. Fukutome et. al. have proposed a new fermion many-body theory based on a $SO(2N+1)$ Lie algebra of fermion operators [7]. A rep of the $SO(2N+1)$ group has been derived by a group extension of a $SO(2N)$ Bogoliubov transformation for fermions to a new canonical transformation group. The fermion Lie operators are represented by bosons [8].

We have extended the SUSY σ -model on the Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$ based on the $SO(2N+1)$ Lie algebra of fermion operators [9]. Following Fukutome, by embedding a $SO(2N+1)$ group into a $SO(2N+2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables [10, 11], we have studied a new aspect of the extended SUSY σ -model and then constructed a Killing potential. Using the coset space $\frac{SO(2N)}{U(N)}$ and constructing the Kähler and Killing potentials, van Holten et. al. have provided the anomaly-free SUSY σ -model on $\frac{SO(2N)}{U(N)}$ [2]. The time dependent (TD) HBT is a powerful tool for describing superconducting fermion systems [3, 4]. The TDHB WF represents the TD behavior of Bose condensate states of fermion pairs. It is a good approximation for the ground state of a fermion system with a pairing interaction, producing the spontaneous Bose condensation. The TDHB equation on the Kähler coset space $\frac{G}{H} = \frac{SO(2N)}{U(N)}$ is derived from the Euler-Lagrange equation of motion for the coset variables. To describe the classical motion on the coset manifold, we start from the local equation of motion [12, 13]. This equation becomes of the [14, 15, 16]. After providing a simple two-level model and a solution for a coset variable, we give a general solution of the TDRHB equation for the coset variables. We obtain the Harish-Chandra decomposition for the $SO(2N)$ matrix based on the nonlinear Möbius transformation together with the geodesic flow on the manifold.

This paper is organized as follows: In §2, we recapitulate briefly the induced representation of the $SO(2N)$ canonical transformation group and the introduction of the $\frac{SO(2N)}{U(N)}$ coset variables. We give a brief sketch of the derivation of the TDHB and the TDRHB equations from the classical Euler-Lagrange equation of motion for the $\frac{SO(2N)}{U(N)}$ coset variables in the TD self-consistent field (TDSCF). In §3, we give the $\frac{SO(2N)}{U(N)}$ Kähler and Killing potentials and present the Harish-Chandra decomposition for the $SO(2N)$ matrix based on the nonlinear Möbius transformation and the Killing potential for tensors. In §4, we provide a simple two-level model and a solution for a coset variable up to the first order and the infinite order in time t . In §5, we give a general solution of the TDRHB equation for the $\frac{SO(2N)}{U(N)}$ coset variables. Finally, in the last section, we give some concluding remarks and further outlook. In the Appendices, after providing the bosonization of $SO(2N)$ Lie operators and vacuum function for bosons, we give another way of derivation of the TDRHB equation. Finally, we give the TDRHB equations for three- and four- level models.

2 Brief summary of $SO(2N)$ Bogoliubov transformation and Derivation of $\frac{SO(2N)}{U(N)}$ Riccati-Hartree-Bogoliubov equation

We give a brief summary of a $SO(2N)$ canonical transformation and of fixing a $\frac{SO(2N)}{U(N)}$ coset variable. Let c_α and c_α^\dagger , ($\alpha=1, \dots, N$), be annihilation and creation operators of the fermion satisfying the canonical anti-commutation relations $\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}$, $\{c_\alpha, c_\beta\} = 0$ and $\{c_\alpha^\dagger, c_\beta^\dagger\} = 0$. We introduce the set of fermion operators consisting of the following pair operators:

$$\left. \begin{aligned} E_\beta^\alpha &= c_\alpha^\dagger c_\beta - \frac{1}{2} \delta_{\alpha\beta}, & E^{\alpha\beta} &= c_\alpha^\dagger c_\beta^\dagger, & E_{\alpha\beta} &= c_\alpha c_\beta, \\ E_\beta^{\alpha\dagger} &= E_\alpha^\beta, & E^{\alpha\beta} &= E_{\beta\alpha}^\dagger, & E_{\alpha\beta} &= -E_{\beta\alpha}. \end{aligned} \right\} \quad (\alpha, \beta = 1, \dots, N) \quad (2.1)$$

It is well known that the set of fermion operators (2.1) form a $SO(2N)$ Lie algebra. As a consequence of the anti-commutation relations, the commutation relations for the fermion operators (2.1) in the $SO(2N)$ Lie algebra are

$$[E_\beta^\alpha, E_\delta^\gamma] = \delta_{\gamma\beta} E_\delta^\alpha - \delta_{\alpha\delta} E_\beta^\gamma, \quad (U(N) \text{ algebra}) \quad (2.2)$$

$$\left. \begin{aligned} [E_\beta^\alpha, E_\gamma^\delta] &= \delta_{\alpha\delta} E_{\beta\gamma} - \delta_{\alpha\gamma} E_{\beta\delta}, \\ [E^{\alpha\beta}, E_\gamma^\delta] &= \delta_{\alpha\delta} E_\gamma^\beta + \delta_{\beta\gamma} E_\delta^\alpha - \delta_{\alpha\gamma} E_\delta^\beta - \delta_{\beta\delta} E_\gamma^\alpha, & [E_{\alpha\beta}, E_\gamma^\delta] &= 0, \end{aligned} \right\} \quad (2.3)$$

We omit the commutation relations obtained by hermitian conjugation of (2.3).

A $SO(2N)$ canonical transformation $U(g)$ is generated by the fermion $SO(2N)$ Lie operators. The transformation $U(g)$ is expressed by a successive transformation as $U(g) = e^\Lambda e^\Gamma$, $\Gamma = \gamma_{\alpha\beta} c_\alpha^\dagger c_\beta$ ($\bar{\gamma}^\dagger = -\gamma$) and $\Lambda = \frac{1}{2} (\lambda_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger + \bar{\lambda}_{\alpha\beta} c_\alpha c_\beta)$ ($\lambda^T = -\lambda$). Introduce matrices g_γ and g_λ as $g_\gamma = \begin{bmatrix} \bar{\gamma} & 0 \\ 0 & \gamma \end{bmatrix}$ and $g_\lambda = \begin{bmatrix} C(\lambda) & \bar{S}(\lambda) \\ S(\lambda) & \bar{C}(\lambda) \end{bmatrix}$. Then the $U(g)$ is the generalized Bogoliubov transformation [3] specified by an $SO(2N)$ matrix g ($\det g = 1$) as,

$$U(g)(c, c^\dagger)U^\dagger(g) = (c, c^\dagger)g, \quad g = g_\lambda g_\gamma \equiv \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix}, \quad \begin{aligned} a &= C(\lambda)\bar{\gamma}, & C(\lambda) &\equiv \cos(\sqrt{\lambda^\dagger \lambda}), \\ b &= S(\lambda)\bar{\gamma}, & S(\lambda) &\equiv \lambda \frac{\sin(\sqrt{\lambda^\dagger \lambda})}{\sqrt{\lambda^\dagger \lambda}}, \end{aligned} \quad (2.4)$$

and $g^\dagger g = g g^\dagger = 1_{2N}$. The $U(g)$ also satisfies the following properties:

$$U(g)U(g') = U(gg'), \quad U(g^{-1}) = U^{-1}(g) = U^\dagger(g), \quad U(1_{2N}) = \mathbb{I}. \quad (2.5)$$

Here, (c, c^\dagger) is a $2N$ -dimensional row vector ($(c_\alpha), (c_\alpha^\dagger)$) and both the $a = (a_\alpha^\alpha)$ and $b = (b_{\alpha\beta})$ are $N \times N$ matrices. The HB ($SO(2N)$) wave function $|g\rangle$ is defined as $|g\rangle = U(g)|0\rangle$ ($|0\rangle$: the vacuum satisfying $c_\alpha|0\rangle = 0$). The wave function, coherent state $|g\rangle$, is expressed as

$$|g\rangle = \langle 0|U(g)|0\rangle \exp\left(\frac{1}{2} q_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger\right)|0\rangle, \quad (2.6)$$

$$\langle 0|U(g)|0\rangle = [\det(a)]^{\frac{1}{2}} = [\det(1_N + q^\dagger q)]^{-\frac{1}{4}} e^{i\frac{\tau}{2}}, \quad \tau = \frac{i}{2} \ln \left[\frac{\det(\bar{a})}{\det(a)} \right], \quad (2.7)$$

$$q = ba^{-1} = \lambda \frac{\tan(\sqrt{\lambda^\dagger \lambda})}{\sqrt{\lambda^\dagger \lambda}}, \quad q = -q^T. \quad (2.8)$$

The q is a variable of the $\frac{SO(2N)}{U(N)}$ coset space. The τ is a phase of the $U(N)$ subgroup and the derivative $\frac{\partial}{\partial \tau}$ appeared in (A.12) in Appendix A plays a crucial role to show the existence of the free fermion vacuum. The function $\overline{\langle 0|U(g)|0\rangle} (\equiv \Phi_{00}(g))$ in $g \in SO(2N)$ corresponds to the free fermion vacuum function as proved in Appendix B. The symbols \det and τ denote the determinant and transposition, respectively. The overline denotes the complex conjugation.

Expectation values of the fermion $SO(2N)$ Lie operators, i.e., the generators of rotation in $2N$ -dimensional Euclidian space, with respect to $|g\rangle$ are given as

$$\left. \begin{aligned} \langle E_{\beta}^{\alpha} + \frac{1}{2}\delta_{\alpha\beta} \rangle_g &= R_{\alpha\beta} = \frac{1}{2}(\bar{b}_{\alpha i} b_{\beta i} - a_i^{\alpha} \bar{a}_i^{\beta}) + \frac{1}{2}\delta_{\alpha\beta} = -[\bar{q}q(1_N - \bar{q}q)^{-1}]_{\alpha\beta}, \\ \langle E_{\alpha\beta} \rangle_g &= -K_{\alpha\beta} = \frac{1}{2}(\bar{a}_i^{\alpha} b_{\beta i} - b_{\alpha i} \bar{a}_i^{\beta}) = -[q(1_N - \bar{q}q)^{-1}]_{\alpha\beta}, \quad \langle E^{\alpha\beta} \rangle_g = \bar{K}_{\alpha\beta}. \end{aligned} \right\} \quad (2.9)$$

The expectation value of a two-body operator is given as

$$\langle E^{\alpha\gamma} E_{\delta\beta} \rangle_g = R_{\alpha\beta} R_{\gamma\delta} - R_{\alpha\delta} R_{\gamma\beta} - \bar{K}_{\alpha\gamma} K_{\delta\beta}. \quad (2.10)$$

Let the Hamiltonian of the fermion system under consideration be

$$H = h_{\alpha\beta} \left(E_{\beta}^{\alpha} + \frac{1}{2}\delta_{\alpha\beta} \right) + \frac{1}{4}[\alpha\beta|\gamma\delta] E^{\alpha\gamma} E_{\delta\beta}. \quad (2.11)$$

The matrix $h_{\alpha\beta}$ related to a single-particle hamiltonian includes a chemical potential and $[\alpha\beta|\gamma\delta] = -[\alpha\delta|\gamma\beta] = [\gamma\delta|\alpha\beta] = [\beta\alpha|\delta\gamma]$ are anti-symmetrized matrix elements of an interaction potential. Parallel to calculations by the usual HB factorization method (see Refs.[4] and [5]), the expectation value of H with respect to $|g\rangle$ is calculated as

$$\begin{aligned} \langle H \rangle_g &= h_{\alpha\beta} \langle E_{\beta}^{\alpha} + \frac{1}{2}\delta_{\alpha\beta} \rangle_g + \frac{1}{2}[\alpha\beta|\gamma\delta] \left\{ \langle E_{\beta}^{\alpha} + \frac{1}{2}\delta_{\alpha\beta} \rangle_g \langle E_{\delta}^{\gamma} + \frac{1}{2}\delta_{\gamma\delta} \rangle_g + \frac{1}{2} \langle E^{\alpha\gamma} \rangle_g \langle E_{\delta\beta} \rangle_g \right\} \\ &= h_{\alpha\beta} R_{\alpha\beta} + \frac{1}{2}[\alpha\beta|\gamma\delta] \left(R_{\alpha\beta} R_{\gamma\delta} - \frac{1}{2} K_{\alpha\gamma}^* K_{\delta\beta} \right). \end{aligned} \quad (2.12)$$

The $SO(2N)$ TDHB equation can be derived from the Euler-Lagrange equation of motion for the $\frac{SO(2N)}{U(N)}$ coset variables q (2.8). We start from the local equations of motion [12, 13] given by

$$\left. \begin{aligned} \dot{q} &= -\frac{i}{\hbar}(1_N - \bar{R})^{-1} \frac{\partial \langle H \rangle_g}{\partial q} (1_N - R)^{-1}, \quad q(1_N - R) = K, \quad \bar{q}K = -R, \\ \dot{\bar{q}} &= -\frac{i}{\hbar}(1_N - R)^{-1} \frac{\partial \langle H \rangle_g}{\partial \bar{q}} (1_N - \bar{R})^{-1}, \quad (1_N - \bar{R})q = K, \quad \bar{K}q = -R. \end{aligned} \right\} \quad (2.13)$$

With the use of the differential formulae,

$$\left. \begin{aligned} \frac{\partial R_{\gamma\delta}}{\partial q_{\alpha\beta}} &= -\bar{K}_{\gamma\alpha}(1_N - R)_{\beta\delta} + \bar{K}_{\gamma\beta}(1_N - R)_{\alpha\delta}, \quad \frac{\partial R_{\gamma\delta}}{\partial \bar{q}_{\alpha\beta}} = -(1_N - R)_{\gamma\alpha} K_{\beta\delta} + (1_N - R)_{\gamma\beta} K_{\alpha\delta}, \\ \frac{\partial K_{\gamma\delta}}{\partial q_{\alpha\beta}} &= (1_N - \bar{R})_{\gamma\alpha}(1_N - R)_{\beta\delta} - (1_N - \bar{R})_{\gamma\beta}(1_N - R)_{\alpha\delta}, \quad \frac{\partial K_{\gamma\delta}}{\partial \bar{q}_{\alpha\beta}} = K_{\gamma\alpha} K_{\beta\delta} - K_{\gamma\beta} K_{\alpha\delta}. \end{aligned} \right\} \quad (2.14)$$

Let us define matrices Q , \mathcal{F}_F and \mathcal{F}_D as $Q \equiv \begin{bmatrix} 0_N & q \\ \bar{q} & 0_N \end{bmatrix}$, $\mathcal{F}_F \equiv \begin{bmatrix} F & 0_N \\ 0_N & -\bar{F} \end{bmatrix}$ and $\mathcal{F}_D \equiv \begin{bmatrix} 0_N & D \\ -\bar{D} & 0_N \end{bmatrix}$.

Then the classical equation of motion for the $\frac{SO(2N)}{U(N)}$ coset variables is calculated to be

$$i\hbar \begin{bmatrix} 0_N & \dot{q} \\ \dot{\bar{q}} & 0_N \end{bmatrix} = \mathcal{F}_D + \mathcal{F}_F Q - Q \mathcal{F}_F - Q \mathcal{F}_D Q = \begin{bmatrix} 0_N & D + Fq + q\bar{F} + q\bar{D}q \\ -\bar{D} - \bar{F}\bar{q} - \bar{q}F - \bar{q}D\bar{q} & 0_N \end{bmatrix}. \quad (2.15)$$

Equations (2.15) are just the Riccati-type matrix equations. Then we call them the $\frac{SO(2N)}{U(N)}$ time dependent Riccati-Hartree-Bogoliubov (TDRHB) equations and $\mathcal{F} (= \mathcal{F}_F + \mathcal{F}_D)$ is the Hartree-Bogoliubov matrix Hamiltonian. The SCF parameters $F = (F_{\alpha\beta}) = F^\dagger$ and $D = (D_{\alpha\beta}) = -D^T$ appeared in the matrix elements in (2.15) are defined by the functional derivatives as

$$F_{\alpha\beta} \equiv \frac{\partial \langle H \rangle_g}{\partial R_{\alpha\beta}} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta] R_{\gamma\delta}, \quad D_{\alpha\beta} \equiv \frac{\partial \langle H \rangle_g}{\partial \bar{K}_{\alpha\beta}} = \frac{1}{2}[\alpha\gamma|\beta\delta](-K_{\delta\gamma}). \quad (2.16)$$

The classical equations of motion on the Kählerian manifold is put into the form (2.14). The Kählerian structure of the symmetric space is carried onto the manifold of coherent state [13].

In the next Section, we study the Kählerian structure through the Kähler potential and the Killing potential.

3 $\frac{SO(2N)}{U(N)}$ Kähler potential and Killing potential

Let us introduce a $2N \times N$ isometric matrix u by $u^T = [b^T, a^T]$. If one uses the matrix elements of u and u^\dagger as the coordinates on the manifold $SO(2N)$, a real line element can be defined by a hermitian metric tensor on the manifold. Under the transformation $u \rightarrow vu$ the metric is invariant. Then the metric tensor defined on the manifold may become singular, due to the fact that one uses too many coordinates.

According to Zumino [1], if a is non-singular, we have relations governing $u^\dagger u$ as

$$\left. \begin{aligned} u^\dagger u &= a^\dagger a + b^\dagger b = a^\dagger \left\{ 1_N + (ba^{-1})^\dagger (ba^{-1}) \right\} a = a^\dagger (1_N + q^\dagger q) a, \\ \ln \det u^\dagger u &= \ln \det (1_N + q^\dagger q) + \ln \det a + \ln \det a^\dagger, \end{aligned} \right\} \quad (3.1)$$

where we have used the $\frac{SO(2N)}{U(N)}$ coset variable q (2.8). If we take the matrix elements of q and \bar{q} as the coordinates on the $\frac{SO(2N)}{U(N)}$ coset manifold, the real line element can be well defined by a hermitian metric tensor on the coset manifold as

$$ds^2 = G_{\alpha\beta\gamma\delta} dq^{\alpha\beta} d\bar{q}^{\gamma\delta} \quad (q^{\alpha\beta} = q_{\alpha\beta} \text{ and } \bar{q}^{\gamma\delta} = \bar{q}_{\gamma\delta}; \quad G_{\alpha\beta\gamma\delta} = G_{\gamma\delta\alpha\beta}). \quad (3.2)$$

We also use the indices $\underline{\gamma}, \underline{\delta}, \dots$ running over α, β, \dots . The condition that the manifold under consideration is a Kähler manifold is that its complex structure is covariantly constant relative to the Riemann connection:

$$G_{\alpha\beta\gamma\delta, \epsilon\varphi} \stackrel{\text{def}}{=} \frac{\partial G_{\alpha\beta\gamma\delta}}{\partial q^{\epsilon\varphi}} = G_{\epsilon\varphi\gamma\delta, \alpha\beta}, \quad G_{\alpha\beta\gamma\delta, \epsilon\varphi} \stackrel{\text{def}}{=} \frac{\partial G_{\alpha\beta\gamma\delta}}{\partial \bar{q}^{\epsilon\varphi}} = G_{\alpha\beta\epsilon\varphi, \gamma\delta}, \quad (3.3)$$

and that it has vanishing torsions. Then, the hermitian metric tensor $G_{\alpha\beta\gamma\delta}$ can be locally given through a real scalar function, the Kähler potential, which takes the well-known form

$$\mathcal{K}(q^\dagger, q) = \ln \det (1_N + q^\dagger q), \quad (3.4)$$

and the explicit expression for the components of the metric tensor is given as

$$G_{\alpha\beta\gamma\delta} = \frac{\partial^2 \mathcal{K}(q^\dagger, q)}{\partial q^{\alpha\beta} \partial \bar{q}^{\gamma\delta}} = \left\{ (1_N + qq^\dagger)^{-1} \right\}_{\delta\alpha} \left\{ (1_N + q^\dagger q)^{-1} \right\}_{\beta\gamma} - (\gamma \leftrightarrow \delta) - (\alpha \leftrightarrow \beta) + (\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta). \quad (3.5)$$

Notice that the above function does not determine the Kähler potential $\mathcal{K}(q^\dagger, q)$ uniquely since the metric tensor $G_{\alpha\beta\gamma\delta}$ is invariant under a transformation of the Kähler potential,

$$\mathcal{K}(q^\dagger, q) \rightarrow \mathcal{K}'(q^\dagger, q) = \mathcal{K}(q^\dagger, q) + \mathcal{F}(q) + \bar{\mathcal{F}}(\bar{q}). \quad (3.6)$$

$\mathcal{F}(q)$ and $\bar{\mathcal{F}}(\bar{q})$ are analytic functions of q and \bar{q} , respectively.

Let us consider a $SO(2N)$ infinitesimal left transformation of a $SO(2N)$ matrix g to g' , $g' = (1_{2N} + \delta g)g$, by using the first equation of (A.3):

$$g' = \begin{bmatrix} 1_N + \delta a & \delta \bar{b} \\ \delta b & 1_N + \delta \bar{a} \end{bmatrix} g = \begin{bmatrix} a + \delta a a + \delta \bar{b} \bar{b} & \bar{b} + \delta a \bar{b} + \delta \bar{b} \bar{a} \\ b + \delta \bar{a} b + \delta b a & \bar{a} + \delta \bar{a} \bar{a} + \delta b \bar{b} \end{bmatrix}. \quad (3.7)$$

If δa and δb satisfy the relations $\delta a^\dagger = -\delta a$, $\text{tr} \delta a = 0$ and $\delta b = -\delta b^T$, the $(1_{2N} + \delta g)$ plays an important role to bosonize the $SO(2N)$ Lie operators as presented in Appendix A. Let us define a $\frac{SO(2N)}{U(N)}$ coset variable $q' (= b'a'^{-1})$ in the g' frame. With the aid of (3.7), the q' is calculated infinitesimally as

$$\begin{aligned} q' &= b'a'^{-1} = (b + \delta \bar{a} b + \delta b a)(a + \delta a a + \delta \bar{b} \bar{b})^{-1} \\ &= q + \delta b - q \delta a + \delta \bar{a} q - q \delta \bar{b} q. \end{aligned} \quad (3.8)$$

The Kähler metrics admit holomorphic isometries (Killing vectors), $\mathcal{R}^{i\alpha}(q)$ and $\bar{\mathcal{R}}^{i\alpha}(\bar{q})$ ($i=1, \dots, \dim g, \alpha=1, \dots, N$). These isometries are solutions of the Killing equation

$$\mathcal{R}^i_{\underline{\beta}}(q), \alpha + \bar{\mathcal{R}}^i_{\alpha}(q), \underline{\beta} = 0, \quad \mathcal{R}^i_{\underline{\beta}}(q) = g_{\alpha\beta} \mathcal{R}^{i\alpha}(q). \quad (3.9)$$

They define infinitesimal symmetry transformations and are described geometrically by the Killing vectors which are generators of infinitesimal coordinate transformations keeping the metric invariant: $\delta q = q' - q = \mathcal{R}(q)$ and $\delta \bar{q} = \bar{\mathcal{R}}(\bar{q})$ such that $g'(q, \bar{q}) = g(q, \bar{q})$. The Killing equation is the necessary and sufficient condition for an infinitesimal coordinate transformation

$$\delta q^\alpha = (\delta b - \delta a^T q - q \delta a + q \delta b^\dagger q)^\alpha = \xi_i \mathcal{R}^{i\alpha}(q), \quad \delta \bar{q}^\alpha = \xi_i \bar{\mathcal{R}}^{i\alpha}(\bar{q}), \quad (3.10)$$

where ξ_i are the infinitesimal and global group parameters. Due to the Killing equation, the Killing vectors $\mathcal{R}^{i\alpha}(q)$ and $\bar{\mathcal{R}}^{i\alpha}(\bar{q})$ can be written locally as the gradient of some real scalar function, the Killing potentials $\mathcal{M}^i(q, \bar{q})$ such that

$$\mathcal{R}^i_{\underline{\alpha}}(q) = -i\mathcal{M}^i_{,\underline{\alpha}}, \quad \bar{\mathcal{R}}^i_{\alpha}(\bar{q}) = i\mathcal{M}^i_{,\alpha}. \quad (3.11)$$

According to van Holten et al. [2] and using the infinitesimal $SO(2N)$ matrix δg given by the first of (A.3), the Killing potential \mathcal{M}_σ can be written for the coset $\frac{SO(2N)}{U(N)}$ as

$$\left. \begin{aligned} \mathcal{M}_\sigma(\delta a, \delta b, \delta b^\dagger) &= \text{Tr}(\delta g \widetilde{\mathcal{M}}_\sigma) = \text{tr}(\delta a \mathcal{M}_{\sigma\delta a} + \delta b \mathcal{M}_{\sigma\delta b^\dagger} + \delta b^\dagger \mathcal{M}_{\sigma\delta b}), \\ \widetilde{\mathcal{M}}_\sigma &\equiv \begin{bmatrix} \widetilde{\mathcal{M}}_{\sigma\delta a} & \widetilde{\mathcal{M}}_{\sigma\delta b^\dagger} \\ -\widetilde{\mathcal{M}}_{\sigma\delta b} & -\widetilde{\mathcal{M}}_{\sigma\delta a^T} \end{bmatrix}, \quad \begin{aligned} \mathcal{M}_{\sigma\delta a} &= \widetilde{\mathcal{M}}_{\sigma\delta a} + (\widetilde{\mathcal{M}}_{\sigma\delta a^T})^T, \\ \mathcal{M}_{\sigma\delta b} &= \widetilde{\mathcal{M}}_{\sigma\delta b}, \quad \mathcal{M}_{\sigma\delta b^\dagger} = \widetilde{\mathcal{M}}_{\sigma\delta b^\dagger}, \end{aligned} \end{aligned} \right\} \quad (3.12)$$

where the trace Tr is taken over the $2N \times 2N$ matrices, while the trace tr is taken over the $N \times N$ matrices. Let us introduce the N -dimensional matrices $\mathcal{R}(q; \delta g)$, $\mathcal{R}_T(q; \delta g)$ and χ by

$$\mathcal{R}(q; \delta g) = \delta b - \delta a^T q - q \delta a + q \delta b^\dagger q, \quad \mathcal{R}_T(q; \delta g) = -\delta a^T + q \delta b^\dagger, \quad \chi = (1_N + q q^\dagger)^{-1} = \chi^\dagger. \quad (3.13)$$

In (3.10), putting $\xi_i = 1$, we have $\delta q = \mathcal{R}(q; \delta g)$ which is the Killing vector in the coset space $\frac{SO(2N)}{U(N)}$, and tr of holomorphic matrix-valued function $\mathcal{R}_T(q; \delta g)$, namely $\text{tr}[\mathcal{R}_T(q; \delta g)] = \mathcal{F}(q)$ is a holomorphic Kähler transformation. Then the Killing potential \mathcal{M}_σ is given as

$$\left. \begin{aligned} -i\mathcal{M}_\sigma(q, \bar{q}; \delta g) &= -\text{tr} \Delta(q, \bar{q}; \delta g), \\ \Delta(q, \bar{q}; \delta g) &\stackrel{\text{def}}{=} \mathcal{R}_T(q; \delta g) - \mathcal{R}(q; \delta g) q^\dagger \chi = (q \delta a q^\dagger - \delta a^T - \delta b q^\dagger + q \delta b^\dagger) \chi. \end{aligned} \right\} \quad (3.14)$$

From (3.12) and (3.14), we obtain

$$-i\mathcal{M}_{\sigma\delta b} = -\chi q, \quad -i\mathcal{M}_{\sigma\delta b^\dagger} = q^\dagger \chi, \quad -i\mathcal{M}_{\sigma\delta a} = 1_N - 2q^\dagger \chi q. \quad (3.15)$$

Using the expression for $\widetilde{\mathcal{M}}_\sigma$, equation (3.15), their components are written in the form

$$-i\widetilde{\mathcal{M}}_{\sigma\delta b} = -\chi q, \quad -i\widetilde{\mathcal{M}}_{\sigma\delta b^\dagger} = q^\dagger \chi, \quad -i\widetilde{\mathcal{M}}_{\sigma\delta a} = -q^\dagger \chi q, \quad -i\widetilde{\mathcal{M}}_{\sigma\delta a^T} = 1_N - q \bar{\chi} q^\dagger = \chi. \quad (3.16)$$

It is easily checked that the result (3.15) satisfies the gradient of the real function \mathcal{M}_σ (3.11). This just the Killing potential \mathcal{M}_σ in the $\frac{SO(2N)}{U(N)}$ coset space obtained by van Holten et al. [2].

To make clear the meaning of the Killing potential, using the $2N \times N$ isometric matrix u ($u^\dagger u = 1_N$), let us introduce the following $2N \times 2N$ matrix:

$$W = uu^\dagger = \begin{bmatrix} R & K \\ -\bar{K} & 1_N - \bar{R} \end{bmatrix}, \quad \begin{aligned} R &= bb^\dagger, \\ K &= ba^\dagger, \end{aligned} \quad (3.17)$$

which satisfies the idempotency relation $W^2 = W$ and is hermitian on the $SO(2N)$ group. The W is the generalized density matrix in the $SO(2N)$ CS rep. Since the matrices a and b are represented in terms of $q = (q_{\alpha\beta})$ as

$$a = (1_N + q^\dagger q)^{-\frac{1}{2}} v, \quad b = q(1_N + q^\dagger q)^{-\frac{1}{2}} v, \quad v \in U(N), \quad (3.18)$$

then, we have

$$R = q(1_N + q^\dagger q)^{-1} q^\dagger = q \bar{\chi} q^\dagger = 1_N - \chi, \quad K = q(1_N + q^\dagger q)^{-1} = \chi q. \quad (3.19)$$

To our great surprise, substituting (3.19) into (3.16), the expression for the Killing potential $-i\widetilde{\mathcal{M}}_\sigma$ just becomes equivalent with the generalized density matrix (3.17). This fact first has been found by the present authors in Ref. [9]. The two relations in (3.18) play an important role to make another way of the derivation of the TDRHB equation as shown in Appendix C.

First according to [2] we define a matrix $\Xi(q)$ and require a transformation rule as follows:

$$\Xi(q) \stackrel{\text{def}}{=} \begin{bmatrix} 1_N & 0 \\ q & 1_N \end{bmatrix}, \quad \Xi^{-1}(q) = \Xi(-q), \quad (3.20)$$

$$\Xi(q) \longrightarrow \Xi({}^q q) = g \Xi(q) \hat{H}^{-1}(q; g), \text{ with } \hat{H}(q; g) = \begin{bmatrix} \left(\hat{H}_+(q; g) \right)^{-1} & \hat{H}_0(q; g) \\ 0 & \hat{H}_-(q; g) \end{bmatrix}. \quad (3.21)$$

It must be remarkable that as is clear from the structure of the transformation (3.21), in the above transformation a constant unitary matrix g can be canceled by taking a bilinear form $\Xi^\dagger({}^q q) \Xi({}^q q)$ [17]. The ${}^q q$ satisfying $({}^q q)^T = {}^q q^T = -{}^q q$ is a nonlinear Möbius transformation given by

$${}^q q = (b + \bar{a}q)(a + \bar{b}q)^{-1} = -(a^T - qb^\dagger)^{-1}(b^T - qa^\dagger), \quad (3.22)$$

which obeys a successive transformation rule $g'({}^q q) = g'{}^q q$, i.e., composition of two transformations g' and g . The above nonlinear Möbius transformation makes a crucial role to construct a solution of the TDRHB equation in the proceeding sections. Under an action of $SO(2N)$ matrix g , i.e., last equation of (2.4), using $g^{-1} = g^\dagger$, (3.21) and (3.22), we have the relation $\hat{H}(q; g) = \Xi(-{}^q q) g \Xi(q)$. Then $\hat{H}(q; g)$ takes a form

$$\hat{H}(q; g) = \begin{bmatrix} a + \bar{b}q & \bar{b} \\ 0 & (a^T - qb^\dagger)^{-1} \end{bmatrix}, \quad \hat{H}_+(q; g) = \hat{H}_-^T(q; g), \quad \det \hat{H}_+(q; g) = \det \hat{H}_-(q; g). \quad (3.23)$$

Multiplying g' by g , we also have useful product formulas for $\hat{H}_{\pm,0}(q; g'g)$ as

$$\hat{H}_+(q; g'g) = \hat{H}_+(q; g) \hat{H}_+({}^q q; g'), \quad \hat{H}_-(q; g'g) = \hat{H}_-({}^q q; g') \hat{H}_-(q; g), \quad \hat{H}_0(q; g'g) = \hat{H}_0({}^q q; g') \hat{H}_0(q; g) = \bar{b}'\bar{b}. \quad (3.24)$$

Then we have the product formula $\hat{H}(q; g'g) = \hat{H}({}^q q; g') \hat{H}(q; g)$.

Using the matrices a and b given by (3.18) ($v = 1_N$), we obtain the matrix decomposition

$$g = \Xi^\dagger(-q) \begin{bmatrix} (1_N + q^\dagger q)^{\frac{1}{2}} & 0 \\ 0 & (1_N + qq^\dagger)^{-\frac{1}{2}} \end{bmatrix} \Xi(q) = \Xi^\dagger \left(-\lambda \frac{\tan(\sqrt{\lambda^\dagger \lambda})}{\sqrt{\lambda^\dagger \lambda}} \right) \begin{bmatrix} \cos^{-1}(\sqrt{\lambda^\dagger \lambda}) & 0 \\ 0 & \cos(\sqrt{\lambda^\dagger \lambda}) \end{bmatrix} \Xi \left(\lambda \frac{\tan(\sqrt{\lambda^\dagger \lambda})}{\sqrt{\lambda^\dagger \lambda}} \right), \quad (3.25)$$

whose relation substituted by the q (2.8) is the Harish-Chandra decomposition. Such a relation is derived based on the nonlinear Möbius transformation (3.22). It should be noticed that the geodesic flow through the identity coset element corresponds to $\lambda \frac{\tan(t\sqrt{\lambda^\dagger \lambda})}{\sqrt{\lambda^\dagger \lambda}}$. These facts have

already been pointed out by Berceanu et. al. [13]. Finally, from the second equation of (3.25), a $SO(2N)$ matrix g is expressed in terms of the original variable λ contained in the generator Λ . For the sake of convenience we redefine the Kähler potential as $\mathcal{K}(q, \bar{q}) = \ln \det (1_N + q\bar{q})$. Under the nonlinear Möbius transformation (3.22), the Kähler potential transforms as $\mathcal{K}({}^q q, {}^q \bar{q}) = \mathcal{K}(q, \bar{q}) + \mathcal{F}(q; g) + \bar{\mathcal{F}}(\bar{q}; g)$.

Next according to [2], to require anomaly cancellations with matter fields, one may change an assignment of $U(1)$ charges by introducing a complex line bundle \mathcal{S} which is defined as a complex matter scalar field coupled to the SUSY σ -model with the infinitesimal transformation law $\delta_i \mathcal{S}^\lambda = \lambda \mathcal{F}_i(q) \mathcal{S}$. For a tensor rep $\mathcal{T}^{\alpha_1 \dots \alpha_p} \equiv \mathcal{S}^\lambda T^{\alpha_1 \dots \alpha_p}$, \mathcal{T} obeys the transformation rule

$$\delta_i \mathcal{T}^{\alpha_1 \dots \alpha_p} = \sum_{k=1}^p \mathcal{R}_{i, \beta}^{\alpha_k}(q) \mathcal{T}^{\alpha_1 \dots \beta \dots \alpha_p} + \lambda \mathcal{F}_i(q) \mathcal{T}^{\alpha_1 \dots \alpha_p}. \quad (3.26)$$

A section of a minimal line bundle over $\frac{SO(2N)}{U(N)}$ is given by ${}^g \mathcal{S} = [\det \hat{H}_+(q; g)]^{\frac{1}{2}} \mathcal{S} = [\det \hat{H}_-(q; g)]^{\frac{1}{2}} \mathcal{S}$.

Suppose that $\mathcal{T}_{(p; q)}^{i_1 \dots i_p}$ is an irreducible completely antisymmetric $SU(N)$ -tensor rep with p and q indices. We abbreviate it simply as $\mathcal{T}_{(p; q)}$. By taking the completely antisymmetric tensor product of $SU(N)$ vectors $\mathcal{T}_1^{i_1}, \dots, \mathcal{T}_p^{i_p}$, we obtain a $SU(N)$ tensor of rank p with index q as

$$\mathcal{T}_{(p; q)}^{i_1 \dots i_p} \equiv \frac{1}{p!} \mathcal{S}^q T_1^{[i_1} * \dots * T_p^{i_p]}, \quad (3.27)$$

where $[\dots]$ denotes the completely anti-symmetrization of the indices inside the brackets.

Thus we obtain a transformation of tensor $\mathcal{T}_{(p; q)}^{i_1 \dots i_p}$ as

$${}^g \mathcal{T}_{(p; q)}^{i_1 \dots i_p} = \left[\det \hat{H}_-(q; g) \right]^{\frac{q}{2}} \left[\hat{H}_-(q; g) \right]_{j_1}^{i_1} \dots \left[\hat{H}_-(q; g) \right]_{j_p}^{i_p} \mathcal{T}_{(p; q)}^{j_1 \dots j_p}. \quad (3.28)$$

The invariant Kähler potential for a tensor is given by

$$\mathcal{K}_{(p;q)} = \overline{\mathcal{T}}_{(p;q)j_1 \dots j_p} \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} \mathcal{T}_{(p;q)}^{i_1 \dots i_p}, \quad \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} \equiv \frac{1}{p!} [\det \chi]^{\frac{q}{2}} \chi^{j_1}_{i_1} \dots \chi^{j_p}_{i_p}. \quad (3.29)$$

A $SU(N)$ dual tensor $\mathcal{T}_{(\overline{N-p};q)i_{p+1} \dots i_N}$ with $(N-p)$ indices and index q is

$$\mathcal{T}_{(\overline{N-p};q)i_{p+1} \dots i_N} \equiv \frac{1}{p!} \mathcal{T}_{(p;q)}^{i_p \dots i_1} \epsilon_{i_1 \dots i_N}, \quad (\epsilon_{i_1 \dots i_N} : SU(N) \text{ Levi-Civita tensor}) \quad (3.30)$$

which transforms under the nonlinear Möbius transformation (3.22) as

$${}^g \mathcal{T}_{(\overline{p};q)i_1 \dots i_p} = \mathcal{T}_{(\overline{p};q)j_1 \dots j_p} \left[\widehat{H}_-^{-1}(q; g) \right]_{i_1}^{j_1} \dots \left[\widehat{H}_-^{-1}(q; g) \right]_{i_p}^{j_p} \left[\det \widehat{H}_-(q; g) \right]^{1+\frac{q}{2}}. \quad (3.31)$$

The invariant Kähler potential for a dual tensor is given by

$$\mathcal{K}_{(\overline{p};q)} = \mathcal{T}_{(\overline{p};q)i_1 \dots i_p} \mathcal{G}_{(\overline{p};q)j_1 \dots j_p}^{i_1 \dots i_p} \overline{\mathcal{T}}_{(\overline{p};q)}^{j_1 \dots j_p}, \quad \mathcal{G}_{(\overline{p};q)j_1 \dots j_p}^{i_1 \dots i_p} \equiv \frac{1}{p!} [\det \chi]^{1+\frac{q}{2}} [\chi^{-1}]^{i_1}_{j_1} \dots [\chi^{-1}]^{i_p}_{j_p}. \quad (3.32)$$

The contributions of the invariant Kähler potentials $\mathcal{K}_{(p;q)}$ and $\mathcal{K}_{(\overline{p};q)}$ to the Killing potentials, $\mathcal{M}_{(p;q)}(q, \overline{q}; \delta g)$ and $\mathcal{M}_{(\overline{p};q)}(q, \overline{q}; \delta g)$ for a tensor $\mathcal{T}_{(\overline{p};q)}$ and a dual tensor $\overline{\mathcal{T}}_{(\overline{p};q)}$ of rank p with index q , are obtained to satisfy $\mathcal{F}_i(q) = 0$ and $\overline{\mathcal{F}}_i(\overline{q}) = 0$ as

$$-i\mathcal{M}_{((\frac{p}{p});q)}(q, \overline{q}; \delta g) = \mathcal{K}_{((\frac{p}{p});q), [\alpha]}(q, \overline{q}; \delta g) \mathcal{R}^{[\alpha]}(q). \quad (3.33)$$

From (3.38) and (3.29), the Killing potential for tensors is exactly computed as

$$\begin{aligned} -i\mathcal{M}_{(p;q)} &= \mathcal{K}_{(p;q), [i]} \mathcal{R}^{[i]} = \overline{\mathcal{T}}_{(p;q)j_1 \dots j_p, [i]} \mathcal{R}^{[i]} \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} \mathcal{T}_{(p;q)}^{i_1 \dots i_p} \\ &+ \overline{\mathcal{T}}_{(p;q)j_1 \dots j_p} \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} \mathcal{R}^{[i]} \mathcal{T}_{(p;q)}^{i_1 \dots i_p} + \overline{\mathcal{T}}_{(p;q)j_1 \dots j_p} \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} \mathcal{T}_{(p;q), [i]}^{i_1 \dots i_p} \mathcal{R}^{[i]}, \quad ([i] = (i\hat{i})), \end{aligned} \quad (3.34)$$

Due to our recent work [18], the variation of $\delta \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p}$ is calculated as

$$\begin{aligned} \delta \mathcal{G}_{(p;q)i_1 \dots i_p}^{j_1 \dots j_p} &= -\frac{q}{2} \frac{1}{p!} [\det \chi]^{\frac{q}{2}} \text{tr} \{ \chi (\delta q \overline{q} + q \delta \overline{q}) \} \chi^{j_1}_{i_1} \dots \chi^{j_p}_{i_p} \\ &- \frac{1}{p!} [\det \chi]^{\frac{q}{2}} \sum_{r=1}^p \chi^{j_1}_{i_1} \dots \{ \chi (\delta q \overline{q} + q \delta \overline{q}) \} \chi^{j_r}_{i_r} \dots \chi^{j_p}_{i_p}, \end{aligned} \quad (3.35)$$

together with the variations

$$\delta \det \chi = -\det \chi \cdot \text{tr} \{ \chi (\delta q \overline{q} + q \delta \overline{q}) \}, \quad \delta \chi^j_i = -\{ \chi (\delta q \overline{q} + q \delta \overline{q}) \} \chi^j_i. \quad (3.36)$$

Taking only the δq term in (3.35), the following type of contraction is easily carried out:

$$\begin{aligned} \mathcal{G}_{(p;q)i_1 \dots i_p, \hat{i}}^{j_1 \dots j_p} \delta q_{\hat{i}\hat{i}} &= -\frac{q}{2} \frac{1}{p!} [\det \chi]^{\frac{q}{2}} \text{tr} (\mathcal{R}_T - \Delta) \chi^{j_1}_{i_1} \dots \chi^{j_p}_{i_p} \\ &- \frac{1}{p!} [\det \chi]^{\frac{q}{2}} \sum_{r=1}^p \chi^{j_1}_{i_1} \dots \left\{ \chi^{j_r}_{i_r} (\mathcal{R}_T - \Delta)^i_{i_r} \right\} \dots \chi^{j_p}_{i_p}, \end{aligned} \quad (3.37)$$

where we have used the relation $\delta q \overline{q} \chi = \mathcal{R}_T - \Delta$. The $(i\hat{i})$ element of the matrix q , i.e., $q_{i\hat{i}}$ denoted as $q^{[i]}$, (\hat{i} : another component different from i) and $\mathcal{R}^{[i]}$ are given by the Killing vector, i.e., $\mathcal{R}^{[i]} = \delta q^{[i]}$, which is the first equation of (3.10) with $\xi_l = 1$. The equation $\overline{\mathcal{T}}_{(p;q)j_1 \dots j_p, [i]} \mathcal{R}^{[i]} = 0$ is evident. Similarly, the Killing potential for dual tensors is also computed. Then we reach

$$\left. \begin{aligned} -i\mathcal{M}_{(p;q)}(q, \overline{q}; \delta \mathcal{G}) &= \frac{1}{p!} \overline{\mathcal{T}}_{(p;q)j_1 \dots j_p} \\ &\times [\det \chi]^{\frac{q}{2}} \chi^{j_1}_{k_1} \dots \chi^{j_p}_{k_p} \\ &\times \left\{ \sum_{r=1}^p \delta^{k_1}_{i_1} \dots [\Delta(q, \overline{q}; \delta \mathcal{G})]^{k_r}_{i_r} \dots \delta^{k_p}_{i_p} + \frac{q}{2} \text{tr} [\Delta(q, \overline{q}; \delta \mathcal{G})] \delta^{k_1}_{i_1} \dots \delta^{k_p}_{i_p} \right\} \mathcal{T}_{(p;q)}^{i_1 \dots i_p}, \\ -i\mathcal{M}_{(\overline{p};q)}(q, \overline{q}; \delta \mathcal{G}) &= \frac{1}{p!} \mathcal{T}_{(\overline{p};q)j_1 \dots j_p} \\ &\times \left\{ \sum_{r=1}^p \delta^{j_1}_{k_1} \dots [-\Delta(q, \overline{q}; \delta \mathcal{G})]^{j_r}_{k_r} \dots \delta^{j_p}_{k_p} + \left(1 + \frac{q}{2}\right) \text{tr} [\Delta(q, \overline{q}; \delta \mathcal{G})] \delta^{j_1}_{k_1} \dots \delta^{j_p}_{k_p} \right\} \\ &\times [\det \chi]^{1+\frac{q}{2}} [\chi^{-1}]^{k_1}_{i_1} \dots [\chi^{-1}]^{k_p}_{i_p} \overline{\mathcal{T}}_{(\overline{p};q)}^{i_1 \dots i_p}, \end{aligned} \right\} \quad (3.38)$$

which is useful to optimize the Killing potential. The λ is a power of complex line bundle \mathcal{S} , \mathcal{S}^λ given by $\lambda = \frac{\mathbf{q}}{2}$ in the upper and $\lambda = 1 + \frac{\mathbf{q}}{2}$ in the lower. The \mathbf{q} stands for a rescaling charge.

4 Two-level model

In this Section, we use a spherical symmetric single-particle state specified by the set of quantum numbers $\{n_a, l_a, j_a, m_a\}$, denoted as α ($\alpha = 1, \dots, N$). The time-reversed single-particle state $\bar{\alpha}$ is obtained from α by changing the sign of m_a . We use a phase factor $s_\alpha = (-1)^{j_a - m_a}$ in the time-reversed quantity. The contribution of the pair interaction to the Hartree-Fock (HF) potential $[\alpha\beta|\gamma\delta]R_{\gamma\delta}$ is neglected. Further assume the pairing potential $D = (D_{\alpha\beta})$ to be constant. This makes the situation very simple as in the BCS theory does [4]. Then the $SO(2N)$ SCF parameters $F_{\alpha\beta}(t)$ and $D_{\alpha\beta}(t)$ defined by (2.16) have the following forms:

$$\left. \begin{aligned} F_{\alpha\beta}(t) &= (\varepsilon_a - \lambda) \cdot \delta_{\alpha\beta}, \\ D_{\alpha\beta}(t) &= -s_\alpha \delta_{\alpha\bar{\beta}} \Delta(t), \quad \Delta(t) \equiv \frac{1}{2} g \sum_\gamma s_\gamma K_{\gamma\bar{\gamma}}(t), \quad s_\gamma K_{\gamma\bar{\gamma}}(t) = s_\gamma [q(1_N - \bar{q}q)^{-1}]_{\gamma\bar{\gamma}}, \end{aligned} \right\} \quad (4.1)$$

where g is the strength parameter for the pairing force.

First we treat a simple two-level model ($N=2$) of a fermion system under consideration. We denote the quantities $q_{12}(t)$ simply as $q(t)$, $(\varepsilon_1 - \lambda) + (\varepsilon_2 - \lambda)$ as 2ε and $D_{12}(t)$ as

$$\left. \begin{aligned} -\mathcal{D}(t) &= s_1 \delta_{1\bar{2}} \Delta(t), \quad \Delta(t) \equiv \frac{1}{2} g (s_1 K_{1\bar{1}}(t) + s_2 K_{2\bar{2}}(t)), \\ s_1 K_{1\bar{1}}(t) &= s_1 \left[\begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \bar{q} \\ -\bar{q} & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix} \right)^{-1} \right]_{1\bar{1}} = s_1 \frac{1}{1+|q|^2} \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}_{1\bar{1}} \\ &= s_1 \frac{1}{1+|q|^2} q, \quad (\text{matrix element: } 1\bar{1}=12), \\ s_2 K_{2\bar{2}}(t) &= s_2 \frac{1}{1+|q|^2} \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}_{2\bar{2}} = -s_2 \frac{1}{1+|q|^2} q, \quad (\text{matrix element: } 2\bar{2}=21), \end{aligned} \right\} \quad (4.2)$$

Then we have the following Riccati equations:

$$\begin{bmatrix} \dot{q}(t) \\ \dot{\bar{q}}(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathcal{D}(t)}{i\hbar} + 2\frac{\varepsilon}{i\hbar} q(t) + \frac{\bar{\mathcal{D}}(t)}{i\hbar} q(t)^2 \\ -\frac{\bar{\mathcal{D}}(t)}{i\hbar} - 2\frac{\varepsilon}{i\hbar} \bar{q}(t) - \frac{\mathcal{D}(t)}{i\hbar} \bar{q}(t)^2 \end{bmatrix}, \quad (4.3)$$

and define the matrix $A(t)$ as

$$A(t) \equiv \begin{bmatrix} \frac{\varepsilon}{i\hbar} & \frac{\mathcal{D}(t)}{i\hbar} \\ -\frac{\bar{\mathcal{D}}(t)}{i\hbar} & -\frac{\varepsilon}{i\hbar} \end{bmatrix}, \quad \det A(t) = \frac{\varepsilon^2}{\hbar^2} - \frac{|D(t)|^2}{\hbar^2} \equiv \delta^2, \quad (4.4)$$

$$A(t)^2 = (-\det A(t)) \cdot 1_2.$$

We give a solution for $q(t)$ up to the first order and the infinite order in t , respectively, as

$$\exp\{tA(t)\} = 1_2 + t \cdot A(t) \quad (\delta^2 = 0), \quad \exp\{tA(t)\} = \frac{\cosh(t\delta)}{\cosh(t\delta)} \cdot 1_2 + \frac{1}{\delta} \frac{\sinh(t\delta)}{\sinh(t\delta)} \cdot A(t) \quad (\delta^2 > 0) \quad (-\delta^2 < 0). \quad (4.5)$$

Following Inoguchi [19], the above solution is shown to satisfy the Riccati equation as follows:

$$\left. \frac{d}{dt} \right|_{t=0} q(T_{\exp\{tA(t)\}}(q)) = \left. \frac{d}{dt} \right|_{t=0} \frac{\cosh(t\delta) + \frac{1}{\delta} \sinh(t\delta) \frac{\varepsilon}{i\hbar} \cdot q + \frac{1}{\delta} \frac{\sinh(t\delta)}{\sinh(t\delta)} \frac{\mathcal{D}(t)}{i\hbar}}{\cosh(t\delta) - \frac{1}{\delta} \sinh(t\delta) \frac{\varepsilon}{i\hbar} - \frac{1}{\delta} \frac{\sinh(t\delta)}{\sinh(t\delta)} \frac{\bar{\mathcal{D}}(t)}{i\hbar} \cdot q} = \frac{\mathcal{D}(t)}{i\hbar} + 2\frac{\varepsilon}{i\hbar} q(t) + \frac{\bar{\mathcal{D}}(t)}{i\hbar} q(t)^2, \quad (4.6)$$

from which, the Riccati equation for $q(t)$ (4.3) can be surely derived. The case $\delta^2=0$ is trivially derived. In Appendix D, we treat three- and four-level models ($N=3$ and 4).

5 General solution of $\frac{\text{SO}(2N)}{\text{U}(N)}$ TDRHB equation

Following the way adopted in the preceding Section, we construct a general solution of time dependent $\frac{\text{SO}(2N)}{\text{U}(N)}$ Riccati-Hartree-Bogoliubov (TDRHB) equation. We here use a modified matrix $\frac{\mathcal{F}}{i\hbar}$ together with $N \times N$ matrices δ^2 and $\bar{\delta}^2$ and suppose relations

$$\frac{\mathcal{F}}{i\hbar} \equiv \begin{bmatrix} \frac{F}{i\hbar} & \frac{D}{i\hbar} \\ -\frac{\bar{D}}{i\hbar} & -\frac{\bar{F}}{i\hbar} \end{bmatrix}, \quad \begin{aligned} -\delta^2 &\equiv \frac{F^2}{(i\hbar)^2} - \frac{D\bar{D}}{(i\hbar)^2}, \quad FD = D\bar{F}, \\ -\bar{\delta}^2 &\equiv \frac{\bar{F}^2}{(i\hbar)^2} - \frac{\bar{D}D}{(i\hbar)^2}, \quad \bar{F}\bar{D} = \bar{D}F. \end{aligned} \quad (5.1)$$

We give a solution for $q(t)$ up to the first order and the infinite order in t , respectively, as

$$\exp\{t\frac{\mathcal{F}}{i\hbar}\} = 1_{2N} + t \cdot \frac{\mathcal{F}}{i\hbar} \quad (||\delta^2|| = 0), \quad (5.2)$$

$$\exp\{t\frac{\mathcal{F}}{i\hbar}\} = \begin{bmatrix} \cos(t\delta) & 0 \\ 0 & \cos(t\bar{\delta}) \end{bmatrix} + \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \bar{\delta}^{-1} \end{bmatrix} \begin{bmatrix} \sin(t\delta) & 0 \\ 0 & \sin(t\bar{\delta}) \end{bmatrix} \cdot \frac{\mathcal{F}}{i\hbar} \quad (||\delta^2||, ||\bar{\delta}^2|| > 0). \quad (5.3)$$

$$\exp\{t\frac{\mathcal{F}}{i\hbar}\} = \begin{bmatrix} \cosh(t\delta) & 0 \\ 0 & \cosh(t\bar{\delta}) \end{bmatrix} + \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \bar{\delta}^{-1} \end{bmatrix} \begin{bmatrix} \sinh(t\delta) & 0 \\ 0 & \sinh(t\bar{\delta}) \end{bmatrix} \cdot \frac{\mathcal{F}}{i\hbar} \quad (-||\delta^2||, -||\bar{\delta}^2|| < 0). \quad (5.4)$$

In the above we suppose the existence of the inverse matrices δ^{-1} and $\bar{\delta}^{-1}$. Following also the Inoguchi method [19], the solutions (5.3) and (5.4) are shown to satisfy the Riccati equation as follows:

$$\begin{aligned} & \cos(t\delta) + \delta^{-1} \sin(t\delta) \frac{F}{i\hbar} \cdot q + \delta^{-1} \sin(t\delta) \frac{D}{i\hbar} \\ \frac{d}{dt} \Big|_{t=0} q(T_{\exp\{tA(t)\}}(q)) &= \frac{d}{dt} \Big|_{t=0} \frac{\cosh(t\delta) + \delta^{-1} \sinh(t\delta) \frac{F}{i\hbar} \cdot q + \delta^{-1} \sinh(t\delta) \frac{D}{i\hbar}}{\cos(t\bar{\delta}) - \bar{\delta}^{-1} \sin(t\bar{\delta}) \frac{\bar{F}}{i\hbar} - \bar{\delta}^{-1} \sin(t\bar{\delta}) \frac{\bar{D}}{i\hbar} \cdot q} = \frac{D}{i\hbar} + \frac{F}{i\hbar} q + q \frac{\bar{F}}{i\hbar} + q \frac{\bar{D}}{i\hbar} q, \quad (5.5) \\ & \cosh(t\bar{\delta}) - \bar{\delta}^{-1} \sinh(t\bar{\delta}) \frac{\bar{F}}{i\hbar} - \bar{\delta}^{-1} \sinh(t\bar{\delta}) \frac{\bar{D}}{i\hbar} \cdot q \end{aligned}$$

from which, we can surely prove that they satisfy the Riccati equation for $q(t)$ (2.15). The case $||\delta^2|| = 0$ (5.2) is also easily proved. The relations $FD = D\bar{F}$ and $\bar{F}\bar{D} = \bar{D}F$ play a crucial role to derive (5.3) and (5.4). They are also satisfied trivially for the previous simple case (4.4).

The above construction of the solution is deeply connected to the way of construction developed by Berceanu, Gheorghe and de Monvel, who have asserted that the matrix Riccati equation is a *flow* on the Grassmann coset manifold [13].

We use a $N \times 2N$ isometric matrix $u = \begin{bmatrix} b \\ a \end{bmatrix}$ and a matrix g given by $g = \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix}$ ($\det g = 1$), given by the first of Eq. (3.7). Let us suppose the following HB eigenvalue equations:

$$\mathcal{F} \begin{bmatrix} b \\ a \end{bmatrix}_i = - \begin{bmatrix} b \\ a \end{bmatrix}_i \epsilon_i, \text{ or } g^T \mathcal{F} = \begin{bmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{bmatrix} g^T, \quad \epsilon \equiv \text{diagonal matrix}(\epsilon_1, \dots, \epsilon_N), \quad \det \mathcal{F} = (-1)^N (\det \epsilon)^2. \quad (5.6)$$

Keeping the form of equation (5.6), we derive a usual TDHB equation in the form as

$$i\hbar \begin{bmatrix} \dot{b} \\ \dot{a} \end{bmatrix} = -\bar{\mathcal{F}} \begin{bmatrix} \bar{b} \\ \bar{a} \end{bmatrix}, \text{ or } i\hbar \begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} = -\bar{\mathcal{F}} \begin{bmatrix} a \\ b \end{bmatrix}, \quad \begin{bmatrix} 0 & 1_N \\ 1_N & 0 \end{bmatrix} (-\bar{\mathcal{F}}) \begin{bmatrix} 0 & 1_N \\ 1_N & 0 \end{bmatrix} = \mathcal{F}, \quad (5.7)$$

which are written in more compact forms as

$$i\hbar \dot{g} = \mathcal{F} g, \text{ or } i\hbar \dot{g}^T = -g^T \mathcal{F}. \quad (5.8)$$

6 Concluding remarks and further outlook

In this paper first we present the induced representation of $SO(2N)$ canonical transformation group and introduce $\frac{SO(2N)}{U(N)}$ coset variables [12]. We give the derivation of the TDHB equation from the Euler-Lagrange equation of motion for the variables in the TDSCF. The TDHB theory is a powerful tool for superconducting fermion systems [3, 4]. The TDHB WF represents the TD behavior of Bose condensate of fermion pairs. It is a good approximation for the ground state of a fermion system with a pairing interaction, producing the spontaneous Bose condensation. The TDHB equation on the Kähler coset space $\frac{G}{H} = \frac{SO(2N)}{U(N)}$ is derived from the Euler-Lagrange equation of motion for the coset variables. To describe the classical on the coset manifold, we start from the local equation of motion [12, 13]. This equation becomes of the Riccati type [14, 15, 16]. After providing a simple two-level model and a solution for a coset variable, we can give successfully a general solution of the TDRHB equation for the coset variables.

Next, along the same strategy developed by van Holten et al. [2], we have extended to the SUSY σ -model on the Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$ based on the $SO(2N+1)$ Lie algebra of fermion operators [9]. Following Fukutome, by embedding a $SO(2N+1)$ group into an anomaly-free spinor rep of $SO(2N+2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables [10, 11], we have studied a new aspect of the extended anomaly-free SUSY σ -model and have given a corresponding Kähler potential and then a Killing potential based on a positive chiral spinor rep. The theory is invariant under a SUSY transformation and the Killing potential is expressed in terms of the coset variables. Using such mathematical manipulation, we have constructed the generalized density matrix in the $SO(2N)$ CS rep and obtained the Harish-Chandra decomposition for the $SO(2N)$ matrix based on the nonlinear Möbius transformation together with the geodesic flow on the manifold. Further using an anomaly-free spinor rep of the $SO(2N)$ group, we have obtained an irreducible completely antisymmetric $SU(N)$ -tensor and -dual tensor under the transformation and then have derived the corresponding Killing potential for such the tensors.

The generalized density matrix $W (=uu^\dagger)$ (3.17) is expressed in term of the matrix g as

$$W = \frac{1}{2}\bar{g} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^T + \frac{1}{2} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}, \quad W^2 = W. \quad (6.1)$$

Using (5.8), the equation of motion for W is given as

$$\begin{aligned} i\hbar\dot{W} &= i\hbar\frac{1}{2}\dot{\bar{g}} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^T + i\hbar\frac{1}{2}\bar{g} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} \dot{g}^T = \mathcal{F}\frac{1}{2}\bar{g} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^T - \mathcal{F}\frac{1}{2}\bar{g} \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^T \\ &= \mathcal{F}\left(W - \frac{1}{2} \cdot 1_{2N}\right) - \left(W - \frac{1}{2} \cdot 1_{2N}\right)\mathcal{F} = [\mathcal{F}, W]. \end{aligned} \quad (6.2)$$

This is just another famous form of the TDHB equation. According to Berceanu et.al. [13], the nonlinear Möbius transformation (3.22) given by $\exp\{t\frac{\mathcal{F}}{i\hbar}\}$ (5.2), (5.3) and (5.4) act on the $\frac{SO(2N)}{U(N)}$ coset variables q and then produces the solution of the TDRHB equation. How does the $\exp\{t\frac{\mathcal{F}}{i\hbar}\}$ act on the generalized density matrix W ? This may be made as follows:

$$\exp\{t\frac{\mathcal{F}}{i\hbar}\}W = \begin{bmatrix} \exp\{t\frac{\mathcal{F}}{i\hbar}\}R & \exp\{t\frac{\mathcal{F}}{i\hbar}\}K \\ -\exp\{t\frac{\mathcal{F}}{i\hbar}\}\bar{K} & \exp\{t\frac{\mathcal{F}}{i\hbar}\}(1_N - \bar{R}) \end{bmatrix}, \quad \begin{aligned} \exp\{t\frac{\mathcal{F}}{i\hbar}\}R &= \exp\{t\frac{\mathcal{F}}{i\hbar}\}(1_N - \chi), \\ \exp\{t\frac{\mathcal{F}}{i\hbar}\}K &= \exp\{t\frac{\mathcal{F}}{i\hbar}\}(\chi q), \end{aligned} \quad \chi = (1_N + qq^\dagger)^{-1}, \quad (6.3)$$

where we have used the relations (3.19). The calculation of $\frac{d}{dt}\bigg|_{t=0} W(T_{\exp\{t\frac{\mathcal{F}}{i\hbar}\}}(W))$ seems to be a very hard task. What is the meaning of this result? Does the result open a new field in theoretical physics? This is an interesting future problem to be solved.

Appendix

A Bosonization of SO(2N) Lie operators

Consider a fermion state vector $|\Psi\rangle$ corresponding to a function $\Psi(g)$ in $g \in SO(2N)$:

$$|\Psi\rangle = \int U(g) |0\rangle \langle 0| U^\dagger(g) |\Psi\rangle dg = \int U(g) |0\rangle \Psi(g) dg. \quad (\text{A.1})$$

The g is given by (3.7) and the dg is an invariant group integration. When an infinitesimal operator $\mathbb{I}_g + \delta\hat{g}$ and a corresponding infinitesimal unitary operator $U(1_{2N} + \delta g)$ is operated on $|\Psi\rangle$, using $U^{-1}(1_{2N} + \delta g) = U(1_{2N} - \delta g)$, it transforms $|\Psi\rangle$ as

$$\begin{aligned} U(1_{2N} - \delta g) |\Psi\rangle &= (\mathbb{I}_g - \delta\hat{g}) |\Psi\rangle = \int U(g) |0\rangle \langle 0| U^\dagger((1_{2N} + \delta g)g) |\Psi\rangle dg \\ &= \int U(g) |0\rangle \Psi((1_{2N} + \delta g)\mathcal{G}) d\mathcal{G} = \int U(g) |0\rangle (1_{2N} + \delta\mathbf{g}) \Psi(g) dg, \end{aligned} \quad (\text{A.2})$$

$$\left. \begin{aligned} 1_{2N} + \delta g &= \begin{bmatrix} 1_N + \delta a & \delta \bar{b} \\ \delta b & 1_N + \delta \bar{a} \end{bmatrix}, \quad \delta a^\dagger = -\delta a, \quad \text{tr} \delta a = 0, \quad \delta b = -\delta b^T, \\ \delta\hat{g} &= \delta a^\alpha_\beta E^\beta_\alpha + \frac{1}{2} (\delta b_{\alpha\beta} E^{\beta\alpha} + \delta \bar{b}_{\alpha\beta} E_{\beta\alpha}), \quad \delta\mathbf{g} = \delta a^\alpha_\beta \mathbf{E}^\beta_\alpha + \frac{1}{2} (\delta b_{\alpha\beta} \mathbf{E}^{\beta\alpha} + \delta \bar{b}_{\alpha\beta} \mathbf{E}_{\beta\alpha}). \end{aligned} \right\} \quad (\text{A.3})$$

Equation (A.2) shows that the operation of $\mathbb{I}_g - \delta\hat{g}$ on the $|\Psi\rangle$ in the fermion space corresponds to the left multiplication by $1_{2N} + \delta g$ for the variable of the g of the function $\Psi(g)$. For a small parameter ϵ , we obtain a representation on the $\Psi(g)$ as

$$\rho(e^{\epsilon\delta g})\Psi(g) = \Psi(e^{\epsilon\delta g}g) = \Psi(g + \epsilon\delta gg) = \Psi(g + dg), \quad (\text{A.4})$$

which leads us to a relation $dg = \epsilon\delta g$. From this, we express it explicitly as,

$$\left. \begin{aligned} dg &= \begin{bmatrix} da & d\bar{b} \\ db & d\bar{a} \end{bmatrix} = \epsilon \begin{bmatrix} \delta aa + \delta \bar{b}b & \delta a\bar{b} + \delta \bar{b}\bar{a} \\ \delta ba + \delta \bar{a}b & \delta \bar{a}\bar{a} + \delta b\bar{b} \end{bmatrix}, \\ da &= \epsilon \frac{\partial a}{\partial \epsilon} = \epsilon(\delta aa + \delta \bar{b}b), \quad db = \epsilon \frac{\partial b}{\partial \epsilon} = \epsilon(\delta ba + \delta \bar{a}b). \end{aligned} \right\} \quad (\text{A.5})$$

A differential representation of $\rho(\delta g)$, $d\rho(\delta g)$, is given as

$$d\rho(\delta g)\Psi(g) = \left[\frac{\partial a^\alpha_\beta}{\partial \epsilon} \frac{\partial}{\partial a^\alpha_\beta} + \frac{\partial b_{\alpha\beta}}{\partial \epsilon} \frac{\partial}{\partial b_{\alpha\beta}} + \frac{\partial \bar{a}^\alpha_\beta}{\partial \epsilon} \frac{\partial}{\partial \bar{a}^\alpha_\beta} + \frac{\partial \bar{b}_{\alpha\beta}}{\partial \epsilon} \frac{\partial}{\partial \bar{b}_{\alpha\beta}} \right] \Psi(g). \quad (\text{A.6})$$

Substituting (A.5) into (A.6), we can get explicit forms of the differential representation

$$d\rho(\delta g)\Psi(g) = \delta\mathbf{g}\Psi(g), \quad (\text{A.7})$$

where each operator in $\delta\mathbf{g}$ is expressed in a differential form as

$$\left. \begin{aligned} \mathbf{E}^\alpha_\beta &= \bar{b}_{\alpha\gamma} \frac{\partial}{\partial \bar{b}_{\beta\gamma}} - b_{\beta\gamma} \frac{\partial}{\partial b_{\alpha\gamma}} - \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{a}^\alpha_\gamma} + a^\alpha_\gamma \frac{\partial}{\partial a^\beta_\gamma} = \mathbf{E}^{\beta\dagger}_\alpha, \\ \mathbf{E}_{\alpha\beta} &= \bar{a}^\alpha_\gamma \frac{\partial}{\partial \bar{b}_{\beta\gamma}} - b_{\beta\gamma} \frac{\partial}{\partial a^\alpha_\gamma} - \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{b}_{\alpha\gamma}} + b_{\alpha\gamma} \frac{\partial}{\partial a^\beta_\gamma} = \mathbf{E}^{\beta\alpha\dagger}, \\ \mathbf{E}^{\alpha\dagger}_\beta &= -\bar{\mathbf{E}}^\alpha_\beta, \quad \mathbf{E}^\dagger_{\alpha\beta} = -\bar{\mathbf{E}}_{\alpha\beta}, \quad \mathbf{E}_{\alpha\beta} = -\mathbf{E}_{\beta\alpha}. \end{aligned} \right\} \quad (\text{A.8})$$

We define the boson operators \mathbf{a}^α_β , $\bar{\mathbf{a}}^\alpha_\beta$, etc., from every variable a^α_β , \bar{a}^α_β , etc., as

$$\left. \begin{aligned} \mathbf{a} &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(a + \frac{\partial}{\partial \bar{a}} \right), \quad \mathbf{a}^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(\bar{a} - \frac{\partial}{\partial a} \right), \quad \bar{\mathbf{a}} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(\bar{a} + \frac{\partial}{\partial a} \right), \quad \mathbf{a}^T \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(a - \frac{\partial}{\partial \bar{a}} \right), \\ [\mathbf{a}, \mathbf{a}^\dagger] &= 1, \quad [\bar{\mathbf{a}}, \mathbf{a}^T] = 1, \quad [\mathbf{a}, \bar{\mathbf{a}}] = [\mathbf{a}, \mathbf{a}^T] = 0, \quad [\mathbf{a}^\dagger, \bar{\mathbf{a}}] = [\mathbf{a}^\dagger, \mathbf{a}^T] = 0, \end{aligned} \right\} \quad (\text{A.9})$$

where a is a complex variable. Similar definitions hold for b in order to define the boson operators $\mathbf{b}_{\alpha\beta}$, $\bar{\mathbf{b}}_{\alpha\beta}$, etc. By noting the relations $\bar{a} \frac{\partial}{\partial \bar{a}} - a \frac{\partial}{\partial a} = \mathbf{a}^\dagger \mathbf{a} - \mathbf{a}^T \bar{\mathbf{a}}$ and $\bar{a} \frac{\partial}{\partial \bar{b}} - b \frac{\partial}{\partial a} = \mathbf{a}^\dagger \mathbf{b} - \mathbf{b}^T \bar{\mathbf{a}}$, the differential operators (A.8) can be converted into a boson operator representation

$$\left. \begin{aligned} \mathbf{E}^\alpha_\beta &= \mathbf{b}^\dagger_{\alpha\gamma} \mathbf{b}_{\beta\gamma} - \mathbf{b}^T_{\beta\gamma} \bar{\mathbf{b}}_{\alpha\gamma} - \mathbf{a}^{\beta\dagger}_\gamma \mathbf{a}^\alpha_\gamma + \mathbf{a}^{pT}_r \bar{\mathbf{a}}^q_r = \mathbf{b}^\dagger_{p\tilde{r}} \mathbf{b}_{q\tilde{r}} - \mathbf{a}^{\beta\dagger}_{\tilde{\gamma}} \mathbf{a}^\alpha_{\tilde{\gamma}}, \\ \mathbf{E}_{\alpha\beta} &= \mathbf{a}^{\alpha\dagger}_\gamma \mathbf{b}_{\beta\gamma} - \mathbf{b}^T_{\beta\gamma} \bar{\mathbf{a}}^\alpha_\gamma - \mathbf{a}^{q\dagger}_\gamma \mathbf{b}_{\alpha\gamma} + \mathbf{b}^T_{\alpha\gamma} \bar{\mathbf{a}}^\beta_\gamma = \mathbf{a}^{\alpha\dagger}_{\tilde{\gamma}} \mathbf{b}_{\beta\tilde{\gamma}} - \mathbf{a}^{\beta\dagger}_{\tilde{\gamma}} \mathbf{b}_{\alpha\tilde{\gamma}}, \end{aligned} \right\} \quad (\text{A.10})$$

by using the notation $\mathbf{a}^{\alpha T}_{\gamma+N} = \mathbf{b}^\dagger_{\alpha\gamma}$ and $\mathbf{b}^T_{\alpha\gamma+N} = \mathbf{a}^{\alpha\dagger}_\gamma$ to use a suffix $\tilde{\gamma}$ running from 1 to N and from N to $2N$. Then we have the boson images of the fermion $SO(2N)$ Lie operators as

$$\mathbf{E}^\alpha_\beta = \mathbf{b}^\dagger_{\alpha\tilde{\gamma}} \mathbf{b}_{\beta\tilde{\gamma}} - \mathbf{a}^{\beta\dagger}_{\tilde{\gamma}} \mathbf{a}^\alpha_{\tilde{\gamma}}, \quad \mathbf{E}_{\alpha\beta} = \mathbf{a}^{\alpha\dagger}_{\tilde{\gamma}} \mathbf{b}_{\beta\tilde{\gamma}} - \mathbf{a}^{\beta\dagger}_{\tilde{\gamma}} \mathbf{b}_{\alpha\tilde{\gamma}}. \quad (\text{A.11})$$

Using the relations $\frac{\partial}{\partial a^\alpha_\beta} \det a = (a^{-1})^\beta_\alpha \det a$ and $\frac{\partial}{\partial a^\alpha_\beta} (a^{-1})^\gamma_\delta = -(a^{-1})^\beta_\delta (a^{-1})^\gamma_\alpha$, we get the relations which are valid when operated on functions on the right coset $\frac{SO(2N)}{SU(N)}$

$$\left. \begin{aligned} \frac{\partial}{\partial b_{\alpha\beta}} &= \sum_{\gamma < \alpha} (a^{-1})^\beta_\gamma \frac{\partial}{\partial q_{\alpha\gamma}}, \\ \frac{\partial}{\partial a^\alpha_\beta} &= -\sum_{\delta < \gamma < \alpha} q_{\gamma\alpha} (a^{-1})^\beta_\delta \frac{\partial}{\partial q_{\gamma\delta}} - \frac{i}{2} (a^{-1})^\beta_\alpha \frac{\partial}{\partial \tau}, \end{aligned} \right\} \quad (\text{A.12})$$

using which, the bosonized operators (A.10) are expressed by the closed first order differential form over the $\frac{SO(2N)}{U(N)}$ coset space in terms of the coset variables $q_{\alpha\beta}$ and a phase variable τ .

B Vacuum function for bosons

We show that function $\Phi_{00}(g) \left(\equiv \overline{\langle 0 | U(g) | 0 \rangle} \right)$ in $g \in SO(2N)$ corresponds to the free fermion vacuum function in the physical fermion space. Then the $\Phi_{00}(g)$ must satisfy the conditions

$$\left(\mathbf{E}^\alpha_\beta + \frac{1}{2} \delta_{\alpha\beta} \right) \Phi_{00}(g) = 0, \quad \Phi_{00}(\mathbf{1}_{2N}) = 1. \quad (\text{B.1})$$

The vacuum function $\Phi_{00}(g)$ which satisfy (B.1) is given by $\Phi_{00}(g) = [\det(\bar{a})]^\frac{1}{2}$, the proof of which is made easily as follows:

$$\begin{aligned} \left(\mathbf{E}^\alpha_\beta + \frac{1}{2} \delta_{\alpha\beta} \right) [\det(\bar{a})]^\frac{1}{2} &= \frac{1}{2} \delta_{\alpha\beta} [\det(\bar{a})]^\frac{1}{2} + \left(\bar{b}_{\alpha\gamma} \frac{\partial}{\partial \bar{b}_{\beta\gamma}} - b_{\beta\gamma} \frac{\partial}{\partial b_{\alpha\gamma}} - \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{a}^\alpha_\gamma} + a^\alpha_\gamma \frac{\partial}{\partial a^\beta_\gamma} \right) [\det(\bar{a})]^\frac{1}{2} \\ &= \frac{1}{2} \delta_{\alpha\beta} [\det(\bar{a})]^\frac{1}{2} - \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{a}^\alpha_\gamma} [\det(\bar{a})]^\frac{1}{2} = \frac{1}{2} \delta_{\alpha\beta} [\det(\bar{a})]^\frac{1}{2} - \frac{1}{2} \frac{1}{[\det(\bar{a})]^\frac{1}{2}} \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{a}^\alpha_\gamma} \det(\bar{a}) \\ &= \frac{1}{2} \delta_{\alpha\beta} [\det(\bar{a})]^\frac{1}{2} - \frac{1}{2} \frac{1}{[\det(\bar{a})]^\frac{1}{2}} (\bar{a} \bar{a}^{-1})_{\beta\alpha} \det(\bar{a}) = 0, \end{aligned} \quad (\text{B.2})$$

$$\mathbf{E}_{\alpha\beta} [\det(\bar{a})]^\frac{1}{2} = \left(\bar{a}^\alpha_\gamma \frac{\partial}{\partial \bar{b}_{\beta\gamma}} - b_{\beta\gamma} \frac{\partial}{\partial a^\alpha_\gamma} - \bar{a}^\beta_\gamma \frac{\partial}{\partial \bar{b}_{\alpha\gamma}} + b_{\alpha\gamma} \frac{\partial}{\partial a^\beta_\gamma} \right) [\det(\bar{a})]^\frac{1}{2} = 0. \quad (\text{B.3})$$

Thus the vacuum function $\Phi_{00}(g)$ in $g \in SO(2N)$ satisfies $\left(\mathbf{E}^\alpha_\beta + \frac{1}{2} \delta_{\alpha\beta} \right) \Phi_{00}(g) = \mathbf{E}_{\alpha\beta} \Phi_{00}(g) = 0$.

C Another way of derivation of $\frac{SO(2N)}{U(N)}$ TD Riccati-Hartree-Bogoliubov equation

As is shown in Section 3, the matrices a and b and χ are represented in terms of $q=(q_{\alpha\beta})$ as

$$\left. \begin{aligned} a &= (1_N + q^\dagger q)^{-\frac{1}{2}} v, \quad b = q(1_N + q^\dagger q)^{-\frac{1}{2}} v, \quad v \in U(N), \\ \chi &\equiv (1_N + q q^\dagger)^{-1}, \quad \bar{\chi} = (1_N + q^\dagger q)^{-1}, \end{aligned} \right\} \quad (C.1)$$

Then the $SO(2N)$ matrix g and the coset variable q are represented as

$$g = \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix} = \begin{bmatrix} a_0 (= \bar{\chi}^{\frac{1}{2}}) & \bar{b}_0 (= \bar{q} \chi^{\frac{1}{2}}) \\ b_0 (= q \bar{\chi}^{\frac{1}{2}}) & \bar{a}_0 (= \chi^{\frac{1}{2}}) \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & \bar{v} \end{bmatrix}, \quad q = b a^{-1} = b_0 a_0^{-1}. \quad (C.2)$$

Along the same strategy as the strategy developed by Chaturvedi et al. [20] and by Fujii and Oike [21], we here give another way of derivation of $\frac{SO(2N)}{U(N)}$ time dependent Riccati-Hartree-Bogoliubov (TDRHB) equation. Let us consider a TD Hamiltonian matrix $\mathcal{H}(t)$, $\mathcal{H}(t) = -\bar{\mathcal{F}}_F(t) - \bar{\mathcal{F}}_D(t) = \begin{bmatrix} -\bar{F}(t) & -\bar{D}(t) \\ D(t) & F(t) \end{bmatrix}$. The unitary evolution operator $g(t)$ is an element of $G = SO(2N)$ obeying the equation

$$i\hbar \dot{g}(t) = \mathcal{H}(t)g(t), \quad g(t_0) = \mathbb{I}. \quad (C.3)$$

Then we have

$$i\hbar \left(\begin{bmatrix} \dot{a}_0 & \dot{\bar{b}}_0 \\ \dot{b}_0 & \dot{\bar{a}}_0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & \bar{v} \end{bmatrix} + \begin{bmatrix} a_0 & \bar{b}_0 \\ b_0 & \bar{a}_0 \end{bmatrix} \begin{bmatrix} \dot{v} & 0 \\ 0 & \dot{\bar{v}} \end{bmatrix} \right) = \begin{bmatrix} -\bar{F}(t) & -\bar{D}(t) \\ D(t) & F(t) \end{bmatrix} \begin{bmatrix} a_0 & \bar{b}_0 \\ b_0 & \bar{a}_0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & \bar{v} \end{bmatrix}, \quad (C.4)$$

which is rewritten as

$$i\hbar \begin{bmatrix} \dot{a}_0 & \dot{\bar{b}}_0 \\ \dot{b}_0 & \dot{\bar{a}}_0 \end{bmatrix} = \begin{bmatrix} -\bar{F}(t) & -\bar{D}(t) \\ D(t) & F(t) \end{bmatrix} \begin{bmatrix} a_0 & \bar{b}_0 \\ b_0 & \bar{a}_0 \end{bmatrix} - i\hbar \begin{bmatrix} a_0 & \bar{b}_0 \\ b_0 & \bar{a}_0 \end{bmatrix} \begin{bmatrix} \dot{v} v^{-1} & 0 \\ 0 & \dot{\bar{v}} \bar{v}^{-1} \end{bmatrix}. \quad (C.5)$$

Equation (C.6) yields

$$\left. \begin{aligned} i\hbar \dot{a}_0 &= -\bar{F}(t)a_0 - \bar{D}(t)b_0 - i\hbar a_0 \dot{v} v^{-1}, \quad i\hbar \dot{b}_0 = F(t)b_0 + D(t)a_0 - i\hbar b_0 \dot{v} v^{-1}, \\ i\hbar \dot{\bar{a}}_0 &= F(t)\bar{a}_0 + D(t)\bar{b}_0 - i\hbar \bar{a}_0 \dot{\bar{v}} \bar{v}^{-1}, \quad i\hbar \dot{\bar{b}}_0 = -\bar{F}(t)\bar{b}_0 - \bar{D}(t)\bar{a}_0 - i\hbar \bar{b}_0 \dot{\bar{v}} \bar{v}^{-1}. \end{aligned} \right\} \quad (C.6)$$

Using the second equation of (C.2) and (C.6),

$$\left. \begin{aligned} i\hbar \dot{q} &= i\hbar \dot{b}_0 a_0^{-1} - i\hbar b_0 a_0^{-1} \dot{a}_0 a_0^{-1} \\ &= D(t) + F(t)q - i\hbar b_0 \dot{v} v^{-1} a_0^{-1} + q \{ \bar{F}(t)a_0 + \bar{D}(t)b_0 + i\hbar a_0 \dot{v} v^{-1} \} a_0^{-1} \\ &= D(t) + F(t)q + q \bar{F}(t) + q \bar{D}(t)q. \end{aligned} \right\} \quad (C.7)$$

This is just the TDRHB equation obtained in (2.15). Note that the relation

$$b_0 a_0^{-1} - b_0 a_0^{-1} \dot{a}_0 a_0^{-1} = \left(\dot{q} \bar{\chi}^{\frac{1}{2}} + \frac{1}{2} q \bar{\chi}^{-\frac{1}{4}} \dot{\bar{\chi}} \bar{\chi}^{-\frac{1}{4}} \right) \bar{\chi}^{-\frac{1}{2}} - q \left(\frac{1}{2} \bar{\chi}^{-\frac{1}{4}} \dot{\chi} \bar{\chi}^{-\frac{1}{4}} \right) \bar{\chi}^{-\frac{1}{2}} = \dot{q}. \quad (C.8)$$

This is consistent with the second equation of (C.2).

D Three- and Four-level models

In this Appendix, we will treat a three-level model ($N=3$). First we denote the quantities q_{12} , q_{13} , q_{23} , $|q^1|^2+|q^2|^2+|q^3|^2$ simply as $q^1(t)$, $q^2(t)$, $q^3(t)$, $|q|^2$, respectively. We also denote $(\varepsilon_1-\lambda)+(\varepsilon_2-\lambda)$, $(\varepsilon_1-\lambda)+(\varepsilon_3-\lambda)$, $(\varepsilon_2-\lambda)+(\varepsilon_3-\lambda)$ as $2\varepsilon^1$, $2\varepsilon^2$, $2\varepsilon^3$ and further D_{12} , D_{13} and D_{23} as $\mathcal{D}^1(t)=-s_1\delta_{1\bar{2}}\Delta(t)$, $\Delta(t)\equiv\frac{1}{2}g(s_1K_{1\bar{1}}(t)+s_2K_{2\bar{2}}(t)+s_3K_{3\bar{3}}(t))$, $\mathcal{D}^2(t)=-s_1\delta_{1\bar{3}}\Delta(t)$ and $\mathcal{D}^3(t)=-s_2\delta_{2\bar{3}}\Delta(t)$, respectively.

The term $s_1K_{1\bar{1}}(t)$ is calculated as

$$\begin{aligned} s_1K_{1\bar{1}}(t) &= s_1 \left[\begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \bar{q}^1 & \bar{q}^2 \\ -\bar{q}^1 & 0 & \bar{q}^3 \\ -\bar{q}^2 & -\bar{q}^3 & 0 \end{bmatrix} \begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix} \right)^{-1} \right]_{1\bar{1}} \\ &= s_1 \left[\begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix} \begin{bmatrix} 1+|q^1|^2+|q^2|^2 & \bar{q}^2q^3 & -\bar{q}^1q^3 \\ \bar{q}^3q^2 & 1+|q^1|^2+|q^3|^2 & \bar{q}^1q^2 \\ -\bar{q}^3q^1 & \bar{q}^2q^1 & 1+|q^2|^2+|q^3|^2 \end{bmatrix}^{-1} \right]_{1\bar{1}} \\ &= s_1 \left[\begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix} \frac{1}{1+|q|^2} \begin{bmatrix} 1+|q^3|^2 & -\bar{q}^2q^3 & \bar{q}^1q^3 \\ -\bar{q}^3q^2 & 1+|q^2|^2 & -\bar{q}^1q^2 \\ \bar{q}^3q^1 & -\bar{q}^2q^1 & 1+|q^1|^2 \end{bmatrix} \right]_{1\bar{1}} = s_1 \frac{1}{1+|q|^2} \begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix}_{1\bar{1}}, \end{aligned} \quad (D.1)$$

where $|q|^2$ is defined as $|q|^2 \equiv |q^1|^2+|q^2|^2+|q^3|^2$. We also have

$$s_2K_{2\bar{2}}(t) = s_2 \frac{1}{1+|q|^2} \begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix}_{2\bar{2}}, \quad s_3K_{3\bar{3}}(t) = s_3 \frac{1}{1+|q|^2} \begin{bmatrix} 0 & q^1 & q^2 \\ -q^1 & 0 & q^3 \\ -q^2 & -q^3 & 0 \end{bmatrix}_{3\bar{3}}, \quad (D.2)$$

Using the above quantities, then we get the following coupled Ricatti equations:

$$i\hbar \begin{bmatrix} \dot{q}^1(t) \\ \dot{q}^2(t) \\ \dot{q}^3(t) \end{bmatrix} = \begin{bmatrix} \mathcal{D}^1(t) + 2\varepsilon^1 q^1(t) + q^1(t)\mathcal{D}^{1\dagger}q^1(t) + q^1(t)\mathcal{D}^{2\dagger}q^2(t) + q^1(t)\mathcal{D}^{3\dagger}q^3(t) \\ \mathcal{D}^2(t) + 2\varepsilon^2 q^2(t) + q^1(t)\mathcal{D}^{1\dagger}q^2(t) + q^2(t)\mathcal{D}^{2\dagger}q^2(t) + q^2(t)\mathcal{D}^{3\dagger}q^3(t) \\ \mathcal{D}^3(t) + 2\varepsilon^3 q^3(t) + q^1(t)\mathcal{D}^{1\dagger}q^3(t) + q^2(t)\mathcal{D}^{2\dagger}q^3(t) + q^3(t)\mathcal{D}^{3\dagger}q^3(t) \end{bmatrix}. \quad (D.3)$$

Further we treat a four-level model ($N=4$). We denote the quantities q_{12} , q_{13} , q_{14} , q_{23} , q_{24} , q_{34} , $|q^1|^2+|q^2|^2+|q^3|^2+|q^4|^2+|q^5|^2+|q^6|^2$ simply as $q^1(t)$, $q^2(t)$, $q^3(t)$, $q^4(t)$, $q^5(t)$, $q^6(t)$, $|q|^2$. Denote $(\varepsilon_1-\lambda)+(\varepsilon_2-\lambda)$, $(\varepsilon_1-\lambda)+(\varepsilon_3-\lambda)$, $(\varepsilon_1-\lambda)+(\varepsilon_4-\lambda)$, $(\varepsilon_2-\lambda)+(\varepsilon_3-\lambda)$, $(\varepsilon_2-\lambda)+(\varepsilon_4-\lambda)$, $(\varepsilon_3-\lambda)+(\varepsilon_4-\lambda)$ as $2\varepsilon^1$, $2\varepsilon^2$, $2\varepsilon^3$, $2\varepsilon^4$, $2\varepsilon^5$, $2\varepsilon^6$ and further D_{12} , D_{13} , D_{14} , D_{23} , D_{24} and D_{34} as

$$\begin{aligned} \mathcal{D}^1(t) &= -s_1\delta_{1\bar{2}}\Delta(t), \quad \mathcal{D}^2(t) = -s_1\delta_{1\bar{3}}\Delta(t), \quad \mathcal{D}^3(t) = -s_1\delta_{1\bar{4}}\Delta(t), \quad \mathcal{D}^4(t) = -s_2\delta_{2\bar{3}}\Delta(t), \\ \mathcal{D}^5(t) &= -s_2\delta_{2\bar{4}}\Delta(t), \quad \mathcal{D}^6(t) = -s_3\delta_{3\bar{4}}\Delta(t) \quad \text{and} \\ \Delta(t) &\equiv \frac{1}{2}g(s_1K_{1\bar{1}}(t)+s_2K_{2\bar{2}}(t)+s_3K_{3\bar{3}}(t)+s_4K_{4\bar{4}}(t)). \end{aligned}$$

Below we use the det which is defined as

$$\begin{aligned} \det &\equiv \left\{ 1 + |q|^2 + |q^1|^2|q^6|^2 + |q^2|^2|q^5|^2 + |q^3|^2|q^4|^2 \right. \\ &\quad \left. - (\bar{q}^2q^3q^4\bar{q}^5 + \text{c.c.}) + (\bar{q}^1q^3q^4\bar{q}^6 + \text{c.c.}) + (\bar{q}^1q^2q^5\bar{q}^6 + \text{c.c.}) \right\}^2. \end{aligned} \quad (D.4)$$

The term $s_1 K_{1\bar{1}}(t)$ is computed as

$$\begin{aligned}
s_1 K_{1\bar{1}}(t) &= s_1 \left[\begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \bar{q}^1 & \bar{q}^2 & \bar{q}^3 \\ -\bar{q}^1 & 0 & \bar{q}^4 & \bar{q}^5 \\ -\bar{q}^2 & -\bar{q}^4 & 0 & \bar{q}^6 \\ -\bar{q}^3 & -\bar{q}^5 & -\bar{q}^6 & 0 \end{bmatrix} \begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} \right)^{-1} \right]_{1\bar{1}} \\
&= s_1 \left[\begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} \begin{bmatrix} 1+|q^1|^2+|q^2|^2+|q^3|^2 & \bar{q}^2 q^4 + \bar{q}^3 q^5 & -\bar{q}^1 q^4 + \bar{q}^3 q^6 & -\bar{q}^1 q^5 - \bar{q}^2 q^6 \\ \bar{q}^4 q^2 + \bar{q}^5 q^3 & 1+|q^1|^2+|q^4|^2+|q^5|^2 & \bar{q}^1 q^2 + \bar{q}^5 q^6 & \bar{q}^1 q^3 - \bar{q}^4 q^6 \\ -\bar{q}^4 q^1 + \bar{q}^6 q^3 & \bar{q}^2 q^1 + \bar{q}^6 q^5 & 1+|q^2|^2+|q^4|^2+|q^6|^2 & \bar{q}^2 q^3 + \bar{q}^4 q^5 \\ -\bar{q}^5 q^1 - \bar{q}^6 q^2 & \bar{q}^3 q^1 - \bar{q}^6 q^4 & \bar{q}^3 q^2 + \bar{q}^5 q^4 & 1+|q^3|^2+|q^5|^2+|q^6|^2 \end{bmatrix} \right]_{1\bar{1}} \quad (D.5) \\
&= s_1 \left[\frac{1}{\det^{\frac{1}{2}}} \begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} \begin{bmatrix} 1+|q^4|^2+|q^5|^2+|q^6|^2 & -\bar{q}^2 q^4 - \bar{q}^3 q^5 & \bar{q}^1 q^4 - \bar{q}^3 q^6 & \bar{q}^1 q^5 + \bar{q}^2 q^6 \\ -\bar{q}^4 q^2 - \bar{q}^5 q^3 & 1+|q^2|^2+|q^3|^2+|q^6|^2 & -\bar{q}^1 q^2 - \bar{q}^5 q^6 & -\bar{q}^1 q^3 + \bar{q}^4 q^6 \\ \bar{q}^4 q^1 - \bar{q}^6 q^3 & -\bar{q}^2 q^1 - \bar{q}^6 q^5 & 1+|q^1|^2+|q^3|^2+|q^5|^2 & -\bar{q}^2 q^3 - \bar{q}^4 q^5 \\ \bar{q}^5 q^1 + \bar{q}^6 q^2 & -\bar{q}^3 q^1 + \bar{q}^6 q^4 & -\bar{q}^3 q^2 - \bar{q}^5 q^4 & 1+|q^1|^2+|q^2|^2+|q^4|^2 \end{bmatrix} \right]_{1\bar{1}} \\
&= s_1 \left[\frac{1}{\det^{\frac{1}{2}}} \begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} + \frac{1}{\det^{\frac{1}{2}}} \begin{bmatrix} 0 & \bar{q}^6 & -\bar{q}^5 & \bar{q}^4 \\ -\bar{q}^6 & 0 & \bar{q}^3 & -\bar{q}^2 \\ \bar{q}^5 & -\bar{q}^3 & 0 & \bar{q}^1 \\ -\bar{q}^4 & \bar{q}^2 & -\bar{q}^1 & 0 \end{bmatrix} (q^1 q^6 - q^2 q^5 + q^3 q^4) \right]_{1\bar{1}},
\end{aligned}$$

We also have

$$s_{2(3)} K_{2\bar{2}(3\bar{3})}(t) = s_{2(3)} \left[\frac{1}{\det^{\frac{1}{2}}} \begin{bmatrix} 0 & q^1 & q^2 & q^3 \\ -q^1 & 0 & q^4 & q^5 \\ -q^2 & -q^4 & 0 & q^6 \\ -q^3 & -q^5 & -q^6 & 0 \end{bmatrix} + \frac{1}{\det^{\frac{1}{2}}} \begin{bmatrix} 0 & \bar{q}^6 & -\bar{q}^5 & \bar{q}^4 \\ -\bar{q}^6 & 0 & \bar{q}^3 & -\bar{q}^2 \\ \bar{q}^5 & -\bar{q}^3 & 0 & \bar{q}^1 \\ -\bar{q}^4 & \bar{q}^2 & -\bar{q}^1 & 0 \end{bmatrix} (q^1 q^6 - q^2 q^5 + q^3 q^4) \right]_{2\bar{2}(3\bar{3})}, \quad (D.6)$$

Then we have the following coupled Ricatti equations:

$$i\hbar \begin{bmatrix} \dot{q}^1(t) \\ \dot{q}^2(t) \\ \dot{q}^3(t) \\ \dot{q}^4(t) \\ \dot{q}^5(t) \\ \dot{q}^6(t) \end{bmatrix} = \begin{bmatrix} \mathcal{D}^1(t) + 2\varepsilon^1 q^1(t) + q^1(t) \mathcal{D}^{1\dagger} q^1(t) + q^1(t) \mathcal{D}^{2\dagger} q^2(t) + q^1(t) \mathcal{D}^{3\dagger} q^3(t) \\ + q^1(t) \mathcal{D}^{4\dagger} q^4(t) + q^1(t) \mathcal{D}^{5\dagger} q^5(t) + q^2(t) \mathcal{D}^{6\dagger} q^5(t) + q^3(t) \mathcal{D}^{6\dagger} q^4(t) \\ \mathcal{D}^2(t) + 2\varepsilon^2 q^2(t) + q^1(t) \mathcal{D}^{1\dagger} q^2(t) + q^2(t) \mathcal{D}^{2\dagger} q^2(t) + q^2(t) \mathcal{D}^{3\dagger} q^3(t) \\ + q^2(t) \mathcal{D}^{4\dagger} q^4(t) + q^3(t) \mathcal{D}^{5\dagger} q^4(t) + q^1(t) \mathcal{D}^{5\dagger} q^6(t) + q^2(t) \mathcal{D}^{6\dagger} q^6(t) \\ \mathcal{D}^3(t) + 2\varepsilon^3 q^3(t) + q^1(t) \mathcal{D}^{1\dagger} q^3(t) + q^2(t) \mathcal{D}^{2\dagger} q^3(t) + q^3(t) \mathcal{D}^{3\dagger} q^3(t) \\ - q^1(t) \mathcal{D}^{4\dagger} q^6(t) + q^2(t) \mathcal{D}^{4\dagger} q^5(t) + q^3(t) \mathcal{D}^{5\dagger} q^5(t) + q^3(t) \mathcal{D}^{6\dagger} q^6(t) \\ \mathcal{D}^4(t) + 2\varepsilon^4 q^4(t) + q^1(t) \mathcal{D}^{1\dagger} q^4(t) + q^2(t) \mathcal{D}^{2\dagger} q^4(t) - q^1(t) \mathcal{D}^{3\dagger} q^6(t) \\ + q^2(t) \mathcal{D}^{3\dagger} q^5(t) + q^4(t) \mathcal{D}^{4\dagger} q^4(t) + q^4(t) \mathcal{D}^{5\dagger} q^5(t) + q^4(t) \mathcal{D}^{6\dagger} q^6(t) \\ \mathcal{D}^5(t) + 2\varepsilon^5 q^5(t) + q^1(t) \mathcal{D}^{1\dagger} q^5(t) + q^1(t) \mathcal{D}^{2\dagger} q^6(t) + q^3(t) \mathcal{D}^{2\dagger} q^4(t) \\ + q^3(t) \mathcal{D}^{3\dagger} q^5(t) + q^4(t) \mathcal{D}^{4\dagger} q^5(t) + q^5(t) \mathcal{D}^{5\dagger} q^5(t) + q^5(t) \mathcal{D}^{6\dagger} q^6(t) \\ \mathcal{D}^6(t) + 2\varepsilon^6 q^6(t) + q^2(t) \mathcal{D}^{1\dagger} q^5(t) - q^3(t) \mathcal{D}^{1\dagger} q^4(t) + q^2(t) \mathcal{D}^{2\dagger} q^6(t) \\ + q^3(t) \mathcal{D}^{3\dagger} q^6(t) + q^3(t) \mathcal{D}^{4\dagger} q^6(t) + q^5(t) \mathcal{D}^{5\dagger} q^6(t) + q^6(t) \mathcal{D}^{6\dagger} q^6(t) \end{bmatrix}. \quad (D.7)$$

Acknowledgements

S. Nishiyama would like to express his sincere thanks to Professor Constança Providência for kind and warm hospitality extended to him at the Centro de Física, Universidade de Coimbra, Portugal. This work was supported by FCT (Portugal) under the project CERN/FP/83505/2008. The authors thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-09-04 on “Development of Quantum Field Theory and String Theory” were useful to complete this work.

References

- [1] B. Zumino, Supersymmetry and Kähler manifolds, Phys. Lett. **87B** (1979) 203-206.
- [2] S. Groot Nibbelink, T.S. Nyawelo and J.W. van Holten, Construction and analysis of anomaly-free supersymmetric $SO(2N)/U(N)$ σ -models, Nucl. Phys. **B594** (2001) 441-476.
- [3] N. N. Bogoliubov, The compensation principle and the self-consistent field method, Soviet Phys. Uspekhi, **67** (1959) 236-254.
- [4] P. Ring and P. Schuck, *The nuclear many-body problem*, Springer, Berlin, 1980.
- [5] J.P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems*, MIT Press, Cambridge, MA, 1986.
- [6] A.M. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, 1986;
Chiral models: Geometrical aspects, Phys. Rep. **146** (1987) 135-213 and references therein.
- [7] H. Fukutome, M. Yamamura and S. Nishiyama, A new fermion many-body theory based on the $SO(2N+1)$ Lie algebra of the fermion operators, Prog. Theor. Phys. **57** (1977) 1554-1571.
- [8] M. Yamamura and S. Nishiyama, An a priori quantized time-dependent Hartree-Bogoliubov theory. - A generalization of the Schwinger representation of quasi-spin to the fermion pair algebra -, Prog. Theor. Phys. **56** (1976) 124-134.
- [9] Seiya Nishiyama, João da Providência, Constança Providência and Flávio Cordeiro, Extended supersymmetric σ -model based on the $SO(2N+1)$ Lie algebra of the fermion operators, Nucl. Phys. **B802** (2008) 121-145.
- [10] H. Fukutome, On the $SO(2N+1)$ regular representation of operators and wave functions of fermion many-body systems, Prog. Theor. Phys. **58** (1977) 1692-1708;
H. Fukutome and S. Nishiyama, Time dependent $SO(2N+1)$ theory for unified description of bose and fermi type collective excitations, Prog. Theor. Phys. **72** (1984) 239-251.
- [11] H. Fukutome, The group theoretical structure of fermion many-body systems arising from the canonical anticommutation relation. I - *Lie algebras of fermion operators and exact generator coordinate representations of state vectors* -, Prog. Theor. Phys. **65** (1981) 809-827.
- [12] S. Nishiyama, Path integral on the coset space of the $SO(2N)$ group and the time-dependent Hartree-Bogoliubov equation, Prog. Theor. Phys. **66** (1981) 348-350.
- [13] V. Ceaurescu and A. Gheorghe, Classical limit and quantization of Hamiltonian systems, in *Symmetries and Semiclassical Features of Nuclear Dynamics*, Invited Lectures of the 1986 International Summer School, Edited by A. A. Raduta, Lecture Notes in Physics, **279**, 69-117, Springer-Verlag Berlin Heidelberg 1987.
S. Berceanu and A. Gheorghe, On equations of motion on complex Grassmann manifold, Rev. Roum. Phys. **36** (1991) 533-554;
On equations of motion on compact Hermitian symmetric spaces, J. Math. Phys. **33** (1992) 998-1007.

- S. Berceanu and L. Boutet de Monvel, Linear dynamical systems, coherent state manifolds, flows, and matrix Riccati equation, *J. Math. Phys.* **34** (1993) 2353-2371.
- [14] Count J.F. Riccati, Animadversiones in aequationes differentiales secundi gradus, *Actarum Eruditorum quae Lipsiae publicantur. Supplementa* **8** (1724) 66-73.
- [15] T.W. Reid, *Riccati Differential Equation*, 16 Mathematics in Science and Engineering, Vol. **86** (Academic, New York, 1972).
- [16] Zakhar-Itkin, The matrix Riccati differential equation and the semi-group of linear fractional transformations, *Russ. Math. Surv.* **28**:3 (1973) 89-131; *Uspekhi Mat. Nauk*, **28**:3 (1973) 83-120.
- [17] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Structure of non-linear realization in supersymmetric theories, *Phys. Lett.* **B** 138 (1984) 94-98;
Non-linear realization in supersymmetric theories, *Prog. Theor. Phys.* **72** (1984) 313-349;
Non-linear realization in supersymmetric theories. II, *ibid* 1207-1213.
M. Bando, T. Kugo and K. Yamawaki, Nonlinear realization and hidden local symmetries, *Phys. Rep.* **164** (1988) 217-314.
- [18] Seiya Nishiyama, João da Providência, Constança Providência and Flávio Cordeiro, Anomaly-free supersymmetric $SO(2N+2)/U(N+1)$ σ -model based on the $SO(2N+1)$ Lie algebra of the fermion operators, *JHEP* **02** (2011) 093-1-093-30.
- [19] Jun-ichi Inoguchi, *Secret of Riccati*, in Japanese, Nihon Hyouron Sha Company, 2010.
- [20] S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda and R. Simon, Ray space 'Riccati' evolution and geometric phases for N -level quantum spaces, *Pramana J. Phys.* **69** (2007) 317-328; arXiv:0706.0964 [quanta-ph].
- [21] K. Fujii and H. Oike, Reduced dynamics from the unitary group to some flag manifolds: Interacting matrix equations, *Int. J. Geom. Methods Mod. Phys.* **6** (2009) 573-581; arXiv:0809.0165 [math-ph].